



Sinyaller ve Sistemler

“Laplace Dönüşümü”

Dr. Cahit Karakuş, 2020

İçerik

- Giriş
- Laplace Transform
- Laplace Transforms of Common Functions
- Properties of Laplace Transforms
- Differentiation & Integration
- Inverse Laplace Transforms
- Initial- and Final-Value Theorems
- Solution of ODE
- Application

Giriş

Why use Laplace Transforms?

- Laplace dönüşümü, Fransız matematikçi tarafından (1749-1827) geliştirildi ve geçen yüzyılda mühendislik problemlerine geniş ölçüde uyarlandı.
- Faydası, diferansiyel denklemleri daha kolay çözülen cebirsel biçimlere dönüştürme yeteneğinde yatmaktadır. Belli alanlarda sistemlerle uğraşmak için bir mühendislik “dili” biçimi olarak çok yaygın hale gelmiştir.
- Cebir kullanarak diferansiyel denkleme çözüm bulunur. Fourier Dönüşümü ile ilişkilidir, sistemlerin davranışlarını, yörüngelerini karakterize etmenin kolay yolunu sunar.
- Diferansiyel denklem çözümlerinde dönüşüme ya da evrişime gerek yoktur. Sistemde birden çok işlem için kullanışlıdır.
- **Fourier dönüşümü, sadece sinüs ve kosinüs temel işlevleri kullanılarak sentezlenebilen sinyalleri analiz etmek için yeterlidir.**
- **Sentez: Bileşenlerden bütünleşik bir sistem oluşturma**
- **Analiz: Bütünleşik bir sistemi bileşenlerine ayrıştırma**
- Sinyalin üstel bileşenleri olduğunda, örneğin, zaman içinde üssel olarak değişkenlik gösteren sinüs dalgası için, fourier'in verdiği sadece yarım bilgidir. Özellikle de başlangıç koşullarına içeren kritik bilgiler kaybolur. Bu tür durumlarda laplace dönüşümü zorunludur.
- Aniden başlayan veya biten herhangi bir şeyi analiz etmek isterseniz, Laplace dönüşümünden büyük ölçüde faydalanırsınız. Bu geçişlere sistem cevabı gibi gerçek uygulamalarda çok karşılaşılr. Aksine, Laplace dönüşümü bir problemi büyük ölçüde basitleştirdiği asimetric fenomenlerle uğraşırken Fourier dönüşümü çok verimsizdir. Laplace'ın çok önemli bir analitik aracı dönüştürmesini sağlar.

Laplace Transformation

- Diferansiyel denklemleri cebirsel ifadelerle çözebilmek
- Kararlılık bölgesini belirlemek

Laplace Dönüşümü

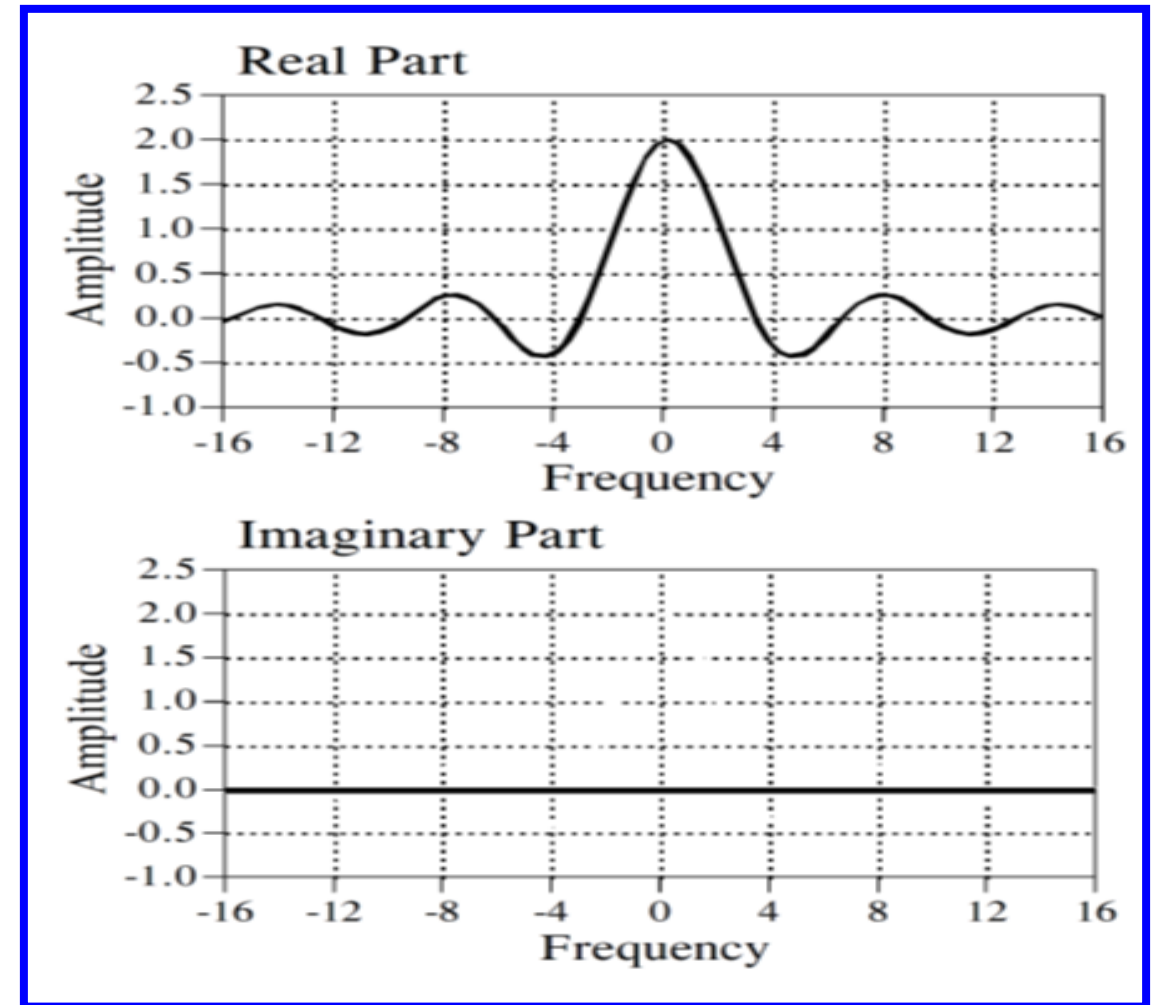
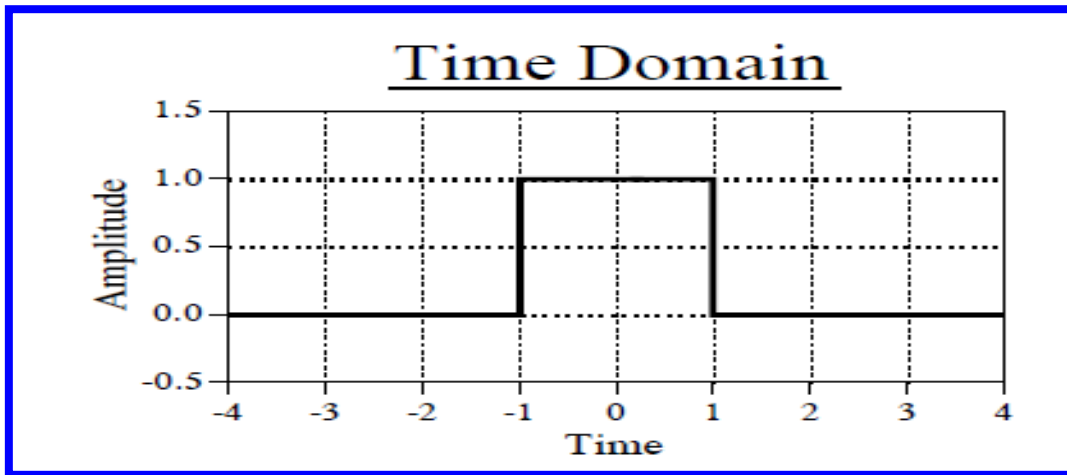
- Doğrusal adi diferansiyel denklemlerin çözümü için önemli bir analitik yöntemdir. Doğrusal olmayan ODE'lerde ilk önce doğrusallaştırılmalıdır. Laplace dönüşümleri önemli proses kontrol kavramları ve tekniklerinde önemli bir rol oynar.
- Sistemlerin, sadece bir frekans veya zaman alanı verileri yerine, sinusoidler ve üsteller gibi doğal bileşenler açısından analiz edilmesine yardımcı olur.
- Yoğun aktarım işlevini, kutuplar ve sıfırlar cinsinden (limit) tanımlanabildiği ve sezgisel olarak sistemin kararlılığını, aşılmasını veya gürültüsünü tahmin etmede yardımcı olan uygun bir alana (S-etki alanı) çevirir.
- Kutuplar: Paydayı sıfır yapan değişkenlerdir.
- Sıfırlar: Payı sıfır yapan değişkenlerdir.
- Fourier Dönüşümü, şifreli bir dönüşümü çok daha kolay bir çarpıma dönüştürürken, bir Laplace Dönüşümü, S-etki alanında basit bir polinom cebiri ile yoğun bir diferansiyel denklemin çözülmesine yardımcı olur.
- Yinelemeli filtrelerin kararlılığını analiz etmek için DSP'de temel olan Z-Dönüşümü, Laplace Dönüşümü'nün yakın akrabasıdır.
- Örnekler: Transfer fonksiyonları, Frekans tepkisi, Kontrol sistemi tasarımı, Kararlılık analizi

Diferansiyel Denklemler Sistemi (Fiziksel Sistemler)

- Sistemi tanımlayan diferansiyel denklemlerin Laplace dönüşümünde, çıktıyı veya ilgili değişkeni çözmek için cebirsel forma dönüştürülür. Çözüm bulmak için ters Laplace dönüşüm uygulanır.
- Systems of Linear Differential Equations
- Basic First-order System Methods
- Structure of Linear Systems
- Matrix Exponential
- The Eigenanalysis Method for $\dot{v} = Ax$
- Jordan Form and Eigenanalysis
- Homogeneous Linear Systems
- Nonhomogeneous Linear Systems
- Second-order Systems
- Numerical Methods for Systems

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \cdot dt$$



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \cdot d\omega$$

Laplace Transformation

$$L[f(t)] = F(s)$$

$$L^{-1}[F(s)] = f(t)$$

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

- t is real, s is complex! $s = \sigma + j\omega$
- Inverse requires complex analysis to solve
- Note “transform”: $f(t) \rightarrow F(s)$, where t is integrated and s is variable
- Conversely $F(s) \rightarrow f(t)$, t is variable and s is integrated
- Assumes $f(t) = 0$ for all $t < 0$

Common transform pairs.

$f(t)$	$F(s) = L[f(t)]$	
1 or $u(t)$	$\frac{1}{s}$	T-1
$e^{-\alpha t}$	$\frac{1}{s + \alpha}$	T-2
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	T-3
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	T-4
$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$	T-5*
$e^{-\alpha t} \cos \omega t$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$	T-6*
t	$\frac{1}{s^2}$	T-7
t^n	$\frac{n!}{s^{n+1}}$	T-8
$e^{-\alpha t} t^n$	$\frac{n!}{(s + \alpha)^{n+1}}$	T-9
$\delta(t)$	1	T-10

*Use when roots are complex.

Constant Function

Constant Function

Let $f(t) = a$ (a constant). Then from the definition of the Laplace transform,

$$\mathcal{L}(a) = \int_0^{\infty} a e^{-st} dt = -\frac{a}{s} e^{-st} \Big|_0^{\infty} = 0 - \left(-\frac{a}{s} \right) = \boxed{\frac{a}{s}}$$

Matlab:

```
clear all; close all  
syms a s t
```

```
f=a
```

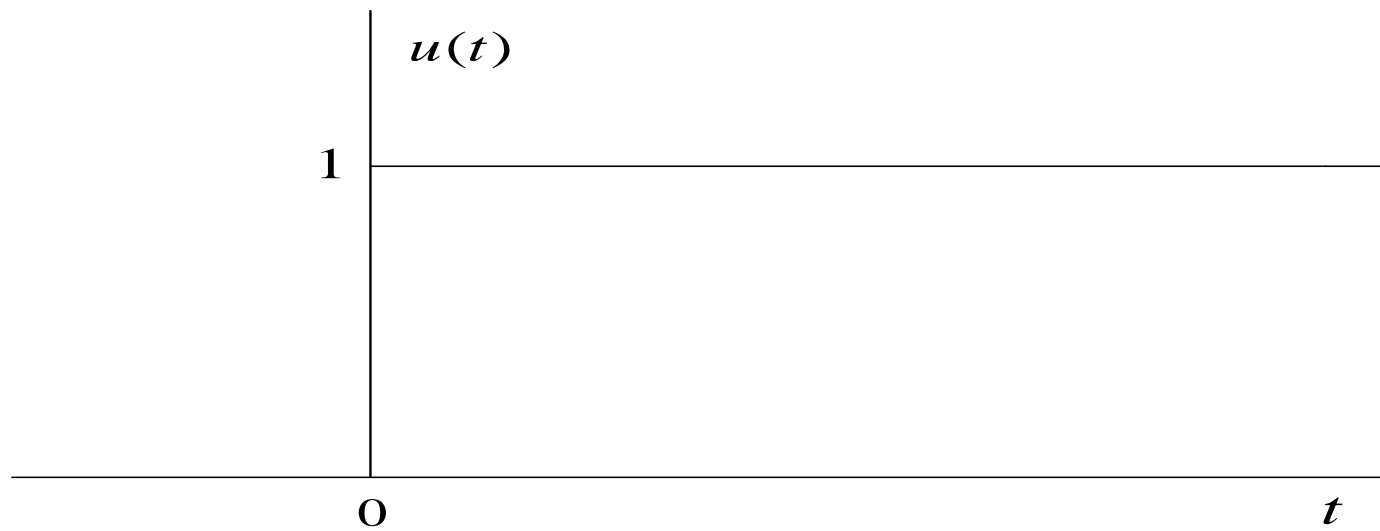
```
F=laplace(f,t,s)
```

Çözüm:

```
f = a
```

```
F = a/s
```

Derive the Laplace transform of the unit step function



$$F(s) = \int_0^{\infty} (1) e^{-st} dt$$

$$F(s) = \left. \frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \left(\frac{e^{-0}}{-s} \right) = \frac{1}{s}$$

$$e^0 = 1$$

$$e^{-0} = \frac{1}{e^0} = 1$$

$$e^{\infty} = \infty$$

$$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

Step Function

Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

$$S(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

Because the step function is a special case of a “constant”,

$$\mathcal{L} [S(t)] = \frac{1}{s}$$

Laplace transform of the exponential function

where $\alpha > 0$. Then,

$$f(t) = e^{-\alpha t}$$

$$F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(\alpha+s)t} dt$$

$$= \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right]_0^{\infty} = 0 - \frac{e^{-0}}{-(s+\alpha)}$$

$$= \frac{1}{s+\alpha}$$

Matlab:

clear all; close all

syms a t

f = exp(-a*t);

F=laplace(f)

Çözüm:

F = 1/(a + s)

Note: Euler's formula

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

$$\begin{aligned} & \frac{1}{s - j\omega_0} & \frac{s}{s^2 + \omega_0^2} & j \frac{\omega_0}{s^2 + \omega_0^2} \\ & = \frac{1}{s - j\omega_0} \frac{s + j\omega_0}{s + j\omega_0} \\ & = \frac{s - j\omega_0}{s^2 + \omega_0^2} \end{aligned}$$

Laplace Transform of $\sin(\beta t)$

$$f(t) = \sin(\beta t) \quad \begin{aligned} e^{j\beta} &= \cos \beta + j \sin \beta \\ e^{-j\beta} &= \cos \beta - j \sin \beta \end{aligned} \quad \begin{aligned} \cos \beta &= \frac{e^{j\beta} + e^{-j\beta}}{2} \\ \sin \beta &= \frac{e^{j\beta} - e^{-j\beta}}{2j} \end{aligned}$$

$$\mathbf{L}[f(t)] = \int_{0^-}^{\infty} \sin(\beta t) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\beta t} - e^{-j\beta t}}{j2} e^{-st} dt = \frac{1}{j2} \left[\int_{0^-}^{\infty} e^{-(s-j\beta)t} dt - \int_{0^-}^{\infty} e^{-(s+j\beta)t} dt \right]$$

If $\text{Re}[s]=\sigma > 0$

$$= \frac{1}{j2} \left(\frac{1}{s-j\beta} - \frac{1}{s+j\beta} \right) = \frac{s+j\beta - s+j\beta}{j2(s^2 + \beta^2)} = \frac{\beta}{s^2 + \beta^2}$$

Laplace Transform of $\cos(\beta t)$

$$f(t) = \cos(\beta t)$$

$$e^{j\beta} = \cos \beta + j \sin \beta$$

$$e^{-j\beta} = \cos \beta - j \sin \beta$$

$$\cos \beta = \frac{e^{j\beta} + e^{-j\beta}}{2}$$

$$\sin \beta = \frac{e^{j\beta} - e^{-j\beta}}{2j}$$

$$\mathbf{L}[f(t)] = \int_{0^-}^{\infty} \cos(\beta t) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \frac{e^{j\beta t} + e^{-j\beta t}}{2} e^{-st} dt = \frac{1}{2} \left[\int_{0^-}^{\infty} e^{-(s-j\beta)t} dt + \int_{0^-}^{\infty} e^{-(s+j\beta)t} dt \right]$$

If $\text{Re}[s]=\sigma>0$

$$= \frac{1}{2} \left(\frac{1}{s-j\beta} + \frac{1}{s+j\beta} \right) = \frac{s+j\beta + s-j\beta}{2(s^2 + \beta^2)} = \frac{s}{s^2 + \beta^2}$$

Summary of Transform Pairs

Time domain \longleftrightarrow s-domain

$$1 \longleftrightarrow \frac{1}{s}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$\sin(\beta t) \longleftrightarrow \frac{\beta}{s^2 + \beta^2}$$

$$\cos(\beta t) \longleftrightarrow \frac{s}{s^2 + \beta^2}$$

Laplace Dönüşümü

- Laplace dönüşümü ile sürekli-zaman Fourier dönüşümü arasındaki ilişki aşağıda gösterilmiştir.

- $s=j\omega$ için,
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \xrightarrow{s=j\omega} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Dolayısıyla ile
$$X(s)\Big|_{s=j\omega} = F\{x(t)\}$$

- $s = \sigma + j\omega$ için,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \xrightarrow{s=\sigma+j\omega} X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}] e^{-j\omega t} dt$$

Bu durumda eşitliğin sağ tarafının $x(t)e^{-\sigma t}$ 'nin Fourier dönüşümüne eşit olduğu görülür.

Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ t - 2, & 2 \leq t. \end{cases}$$

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^2 e^{-st} dt + \int_2^{\infty} (t - 2)e^{-st} dt \\ &= \frac{1}{-s} e^{-st} \Big|_{t=0}^2 + (t - 2) \frac{1}{-s} e^{-st} \Big|_{t=2}^{\infty} - \int_2^{\infty} \frac{1}{-s} e^{-st} dt \\ &= \frac{1}{-s} (e^{-2s} - 1) + (0 - 0) + \frac{1}{s} \frac{1}{-s} e^{-st} \Big|_{t=2}^{\infty} = \frac{1}{-s} (e^{-2s} - 1) + \frac{1}{s^2} e^{-2s} \end{aligned}$$

Example. A force in newtons (N) is given below.
Determine the Laplace transform.

$$f(t) = 50u(t)$$

$$F(s) = \frac{50}{s}$$

Example. A pressure in pascals (p) starting at $t = 0$ is given below. Determine the Laplace transform.

$$p(t) = 5 \cos 2t + 3e^{-4t}$$

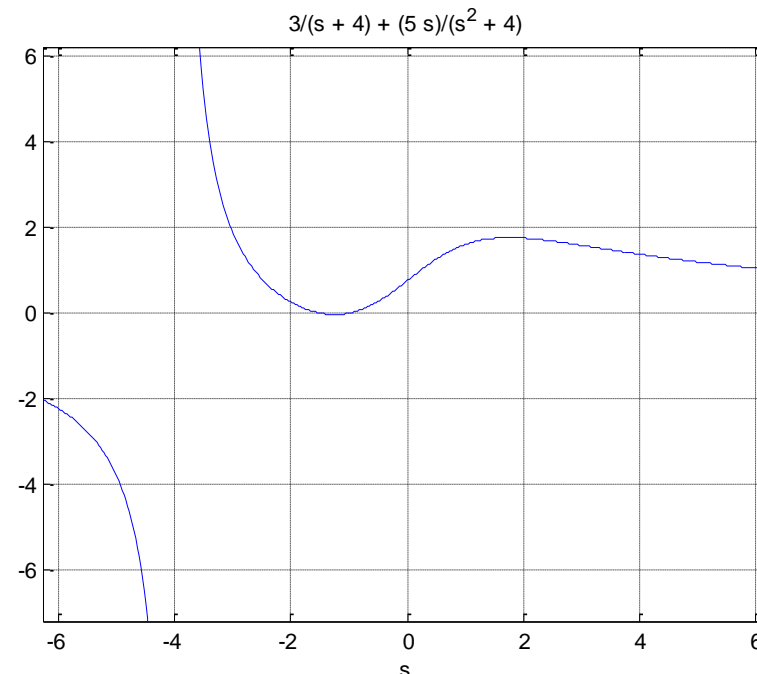
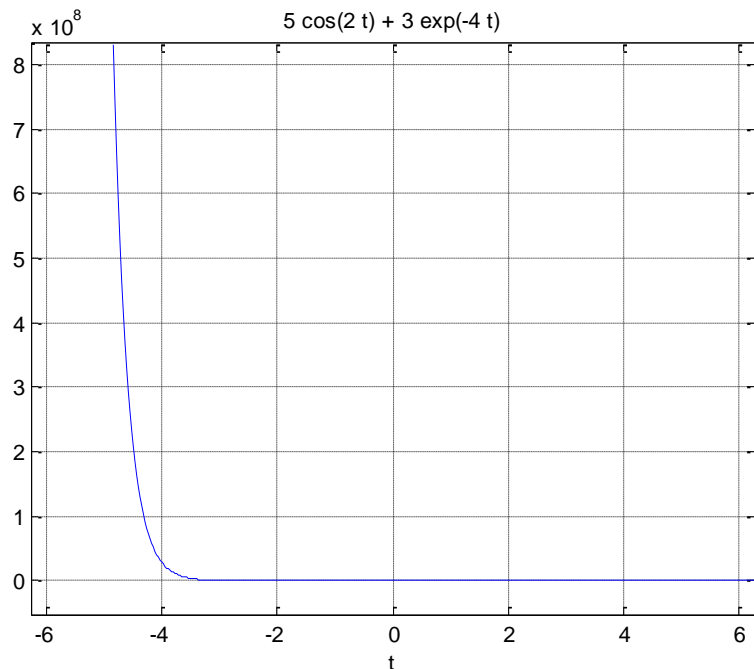
$$P(s) = L[p(t)] = 5 \cdot \frac{s}{s^2 + (2)^2} + 3 \cdot \frac{1}{s + 4}$$

$$= \frac{5s}{s^2 + 4} + \frac{3}{s + 4}$$

Determine the Laplace transform.

$$p(t) = 5 \cos 2t + 3 e^{-4t}$$

$$\begin{aligned} P(s) &= L[p(t)] = 5 \cdot \frac{s}{s^2 + (2)^2} + 3 \cdot \frac{1}{s + 4} \\ &= \frac{5s}{s^2 + 4} + \frac{3}{s + 4} \end{aligned}$$



Matlab:

clear all; close all

syms a t

f=5*cos(2*t)+3*exp(-4*t);

F=laplace(f)

F1=simplify(F)

figure, ezplot(f), grid on

figure, ezplot(F), grid on

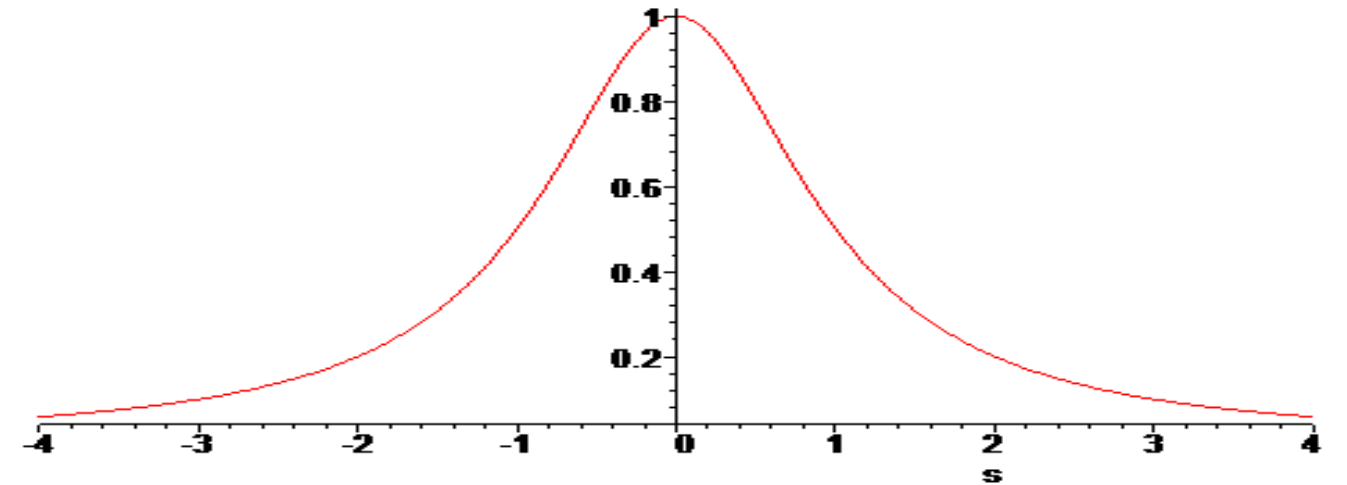
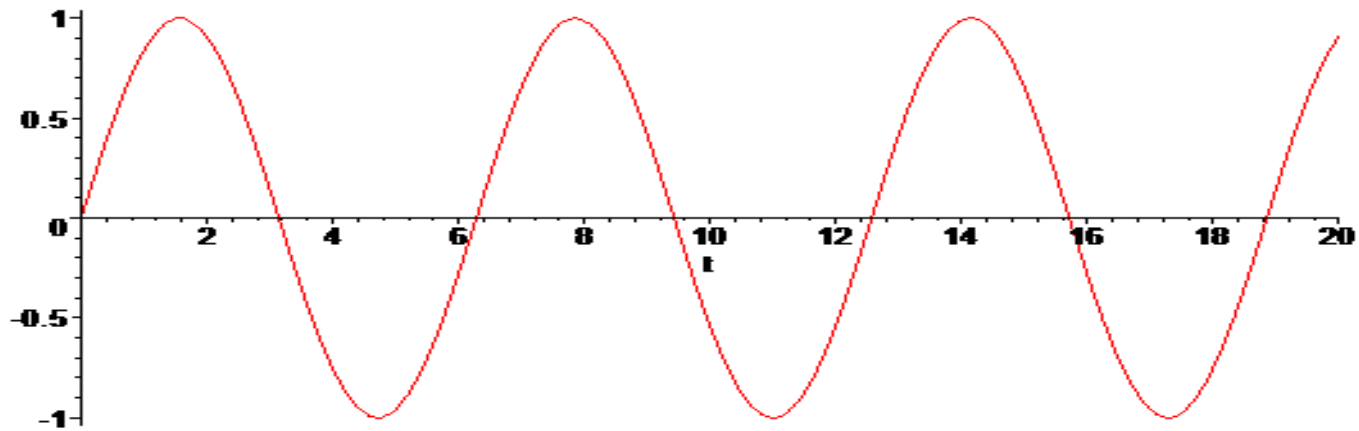
Çözüm:

$$F = 3/(s + 4) + (5*s)/(s^2 + 4)$$

Laplace Transform Time-Scaling Property

- Consider $\sin(t)$

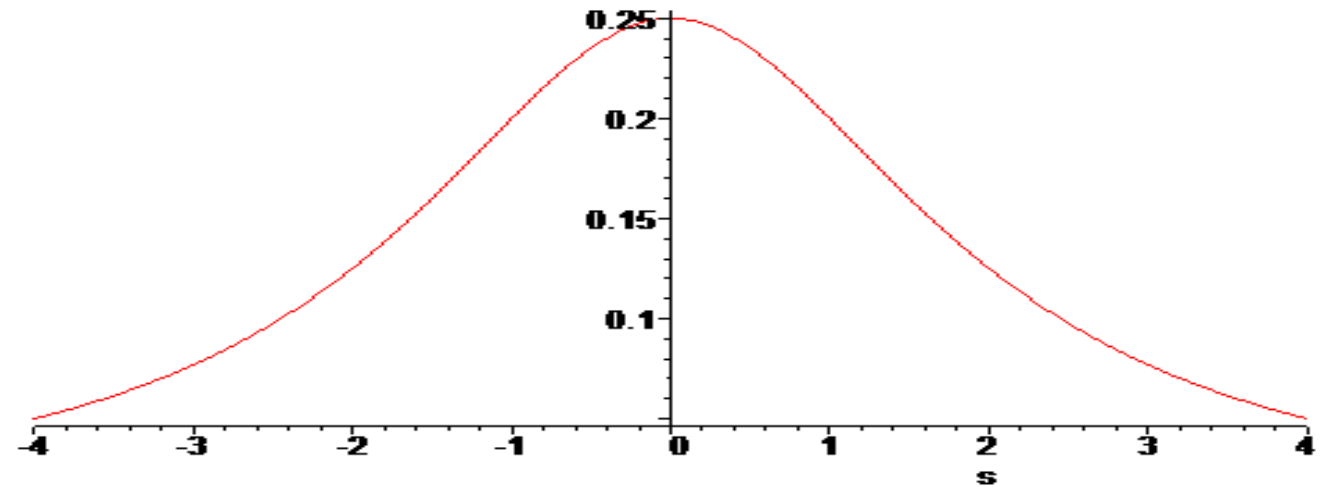
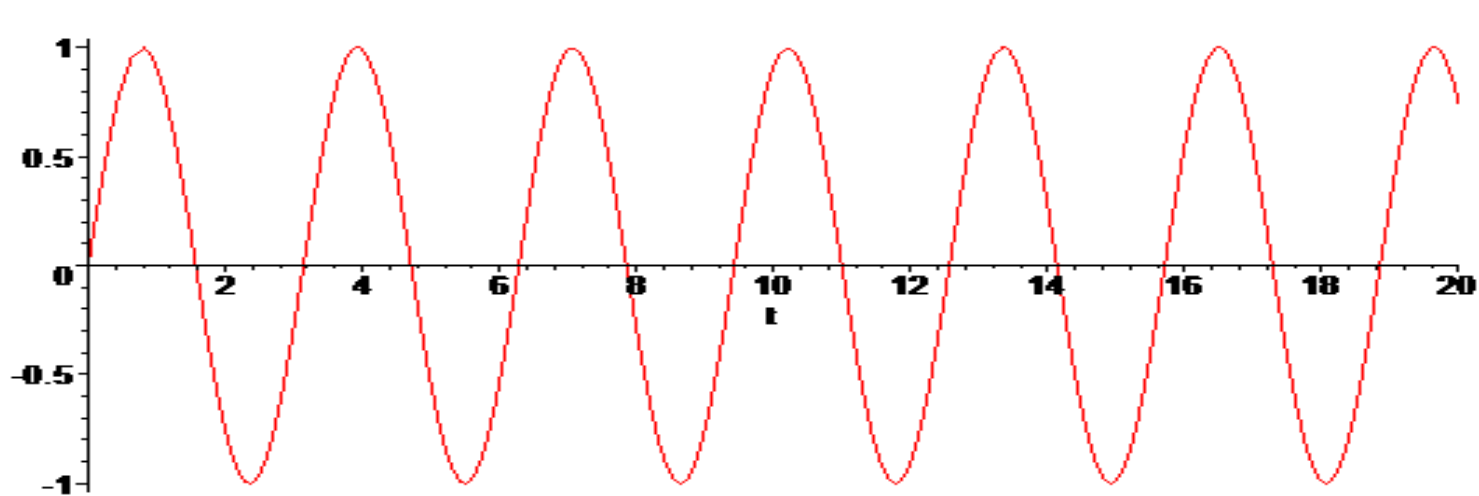
$$\sin(t) \Leftrightarrow \frac{1}{s^2 + 1}$$



Laplace Transform Time-Scaling Property

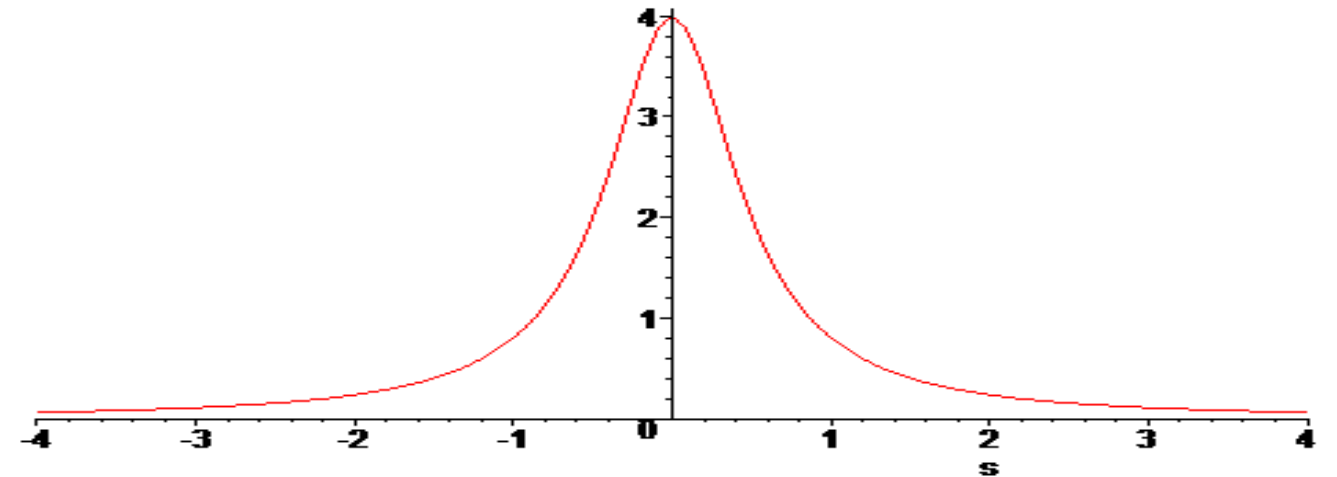
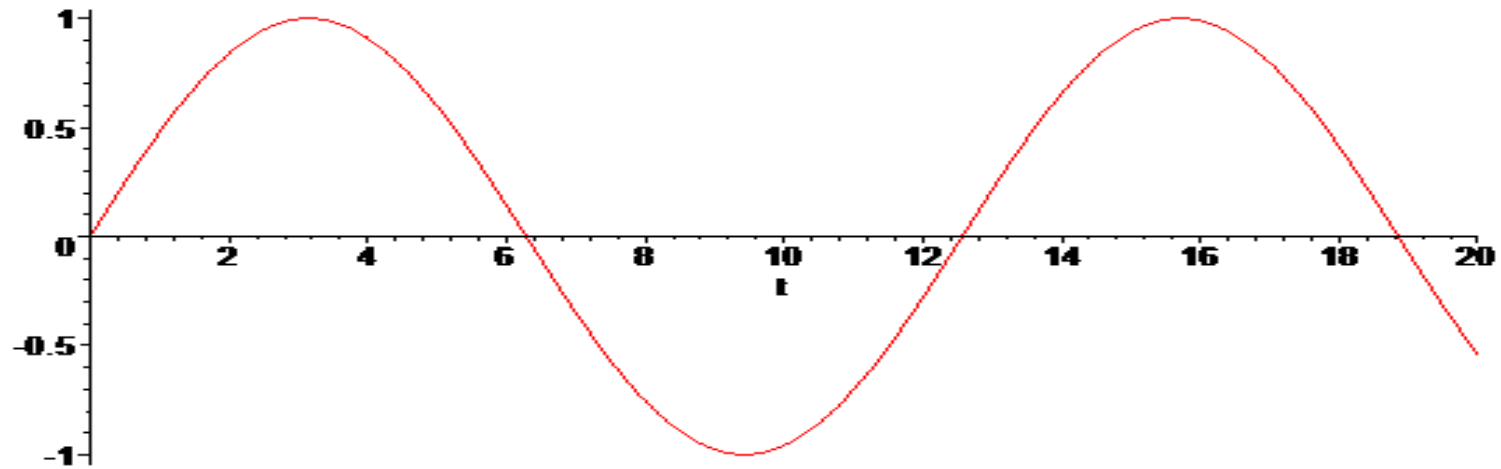
- Time scale by $\omega = 2$

$$\sin(2t) \Leftrightarrow \frac{1}{s^2 + 2^2}$$



Laplace Transform Time-Scaling Property

- Time scale by $\omega = 1/2$ $\sin(1/2 t) \Leftrightarrow \frac{1}{s^2 + \frac{1}{4}}$



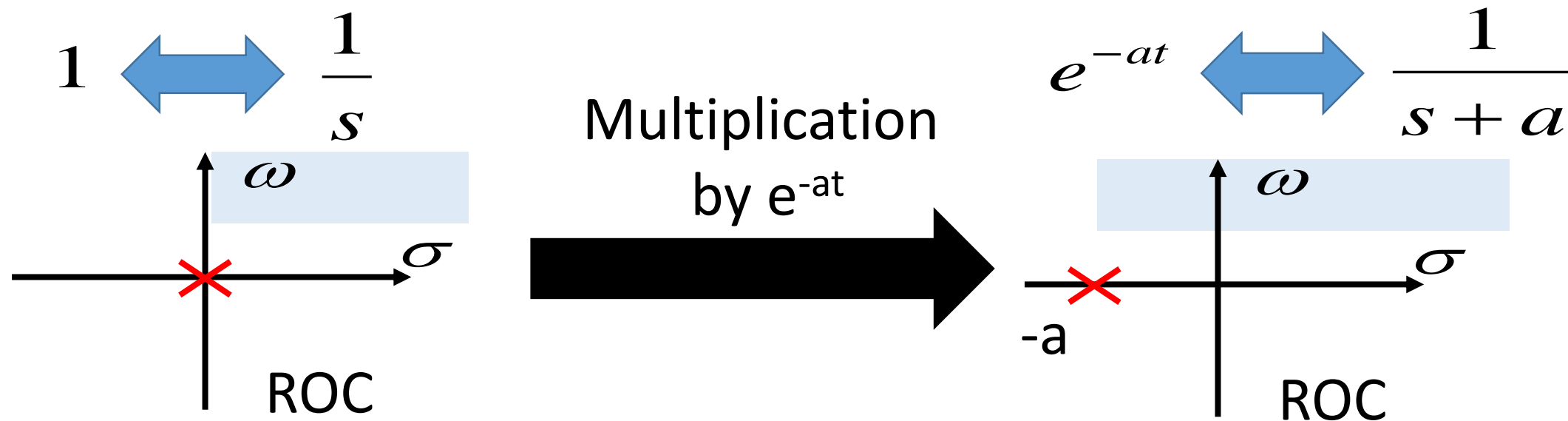
Properties of Laplace Transforms

Linear Combination

$$\begin{aligned}L[f(t) + g(t)] &= \int_{0^-}^{\infty} [f(t) + g(t)]e^{-st} dt \\ &= \int_{0^-}^{\infty} f(t)e^{-st} dt + \int_{0^-}^{\infty} g(t)e^{-st} dt \\ &= F(s) + G(s)\end{aligned}$$

Multiplication by e^{-at}

$$\begin{aligned}\mathbf{L}[e^{-at} f(t)] &= \int_0^{\infty} e^{-at} f(t) \cdot e^{-st} dt \\ &= \int_0^{\infty} f(t) \cdot e^{-(s+a)t} dt = F(s+a)\end{aligned}$$



Example: A voltage in volts (V) starting at $t = 0$ is given below.
Determine the Laplace transform.

$$v(t) = 5e^{-2t} \sin 4t$$

$$\begin{aligned} V(s) &= L[v(t)] = 5 \cdot \frac{4}{(s+2)^2 + (4)^2} \\ &= \frac{20}{s^2 + 4s + 4 + 16} = \frac{20}{s^2 + 4s + 20} \end{aligned}$$

Matlab:

```
clear all; close all
```

```
syms a t
```

```
f=5*sin(4*t)*exp(-2*t);
```

```
F=laplace(f)
```

```
F1=simplify(F)
```

```
F2= expand(F)
```

Çözüm:

```
F =20/((s + 2)^2 + 16)
```

```
F1 =20/((s + 2)^2 + 16)
```

```
F2 =20/(s^2 + 4*s + 20)
```

Multiplication by e^{at}

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}, \quad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}.$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}, \quad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2}.$$

Example

Example

$$\mathcal{L}\{(t^2 + 4)e^{2t} - e^{-t} \cos t\} = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2 + 1},$$

because

$$\mathcal{L}\{t^2 + 4\} = \frac{2}{s^3} + \frac{4}{s}, \quad \Rightarrow \mathcal{L}\{(t^2 + 4)e^{2t}\} = \frac{2}{(s-2)^3} + \frac{4}{s-2}.$$

Matlab:

```
clear all; close all
```

```
syms a t
```

```
f=(t^2+4)*exp(2*t)-cos(t)*exp(-t);
```

```
F=laplace(f)
```

```
F1=simplify(F)
```

Çözüm:

$$F = 4/(s-2) - (s+1)/((s+1)^2 + 1) + 2/(s-2)^3$$

Inverse Laplace Transforms

Inverse Laplace Transforms

- Bir diferansiyel denklem Laplace dönüşümleri ile çözüldüğünde, çözüm s değişkeninin bir fonksiyonu olarak elde edilir. Zaman tepkisini belirlemek için ters dönüşüm oluşturulmalıdır. En basit formlar, tablolarla tanınabilen formlardır ve şimdi bunlardan birkaçı ele alınacaktır.

Inverse Laplace Transforms

Background:

To find the inverse Laplace transform we use transform pairs along with partial fraction expansion:
F(s) can be written as;

$$F(s) = \frac{P(s)}{Q(s)}$$

Where P(s) & Q(s) are polynomials in the Laplace variable, s. We assume the order of Q(s) \geq P(s), in order to be in proper form. If F(s) is not in proper form we use long division and divide Q(s) into P(s) until we get a remaining ratio of polynomials that are in proper form.

Inverse Laplace Transforms

There are three cases to consider in doing the partial fraction expansion of $F(s)$.

Case 1: $F(s)$ has all non repeated simple roots.

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \dots + \frac{k_n}{s + p_n}$$

Case 2: $F(s)$ has complex poles:

$$F(s) = \frac{P_1(s)}{Q_1(s)(s + \alpha - j\beta)(s + \alpha + j\beta)} = \frac{k_1}{s + \alpha - j\beta} + \frac{k_1^*}{s + \alpha + j\beta} + \dots + \text{(expanded)}$$

Case 3: $F(s)$ has repeated poles.

$$F(s) = \frac{P_1(s)}{Q_1(s)(s + p_1)^r} = \frac{k_{11}}{s + p_1} + \frac{k_{12}}{(s + p_1)^2} + \dots + \frac{k_{1r}}{(s + p_1)^r} + \dots + \frac{P_1(s)}{Q_1(s)} \text{(expanded)}$$

Determine the inverse transform of the function below.

$$F(s) = \frac{5}{s} + \frac{12}{s^2} + \frac{8}{s+3}$$

$$f(t) = 5 + 12t + 8e^{-3t}$$

Matlab:

```
clear all; close all
```

```
syms a s
```

```
F=5/s+12/s^2+8/(s+3);
```

```
f=ilaplace(f)
```

Çözüm:

```
f=12*t + 8*exp(-3*t) + 5
```

Determine the inverse transform of the function below.

$$V(s) = \frac{200}{s^2 + 100}$$

$$V(s) = 20 \left(\frac{10}{s^2 + (10)^2} \right)$$

$$v(t) = 20 \sin 10t$$

Matlab:

```
clear all; close all
```

```
syms a s
```

```
f=200/(s^2+100);
```

```
F=ilaplace(f)
```

Çözüm:

```
F=20*sin(10*t)
```

Example. Determine the inverse transform of the function.

$$V(s) = \frac{8s + 4}{s^2 + 6s + 13} \quad s_{1,2} = -3 \pm 2i$$

When the denominator contains a quadratic, check the roots. If they are real, a partial fraction expansion will be required. If they are complex, the table may be used. In this case, the roots are

$$\begin{aligned} s^2 + 6s + 13 \\ &= s^2 + 6s + (3)^2 + 13 - (3)^2 \\ &= s^2 + 6s + 9 + 4 \\ &= (s + 3)^2 + (2)^2 \end{aligned}$$

$$\begin{aligned} V(s) &= \frac{8(s + 3)}{(s + 3)^2 + (2)^2} + \frac{4 - 24}{(s + 3)^2 + (2)^2} \\ &= \frac{8(s + 3)}{(s + 3)^2 + (2)^2} - \frac{10(2)}{(s + 3)^2 + (2)^2} \end{aligned}$$

$$v(t) = 8e^{-3t} \cos 2t - 10e^{-3t} \sin 2t$$

Example: Determine the inverse transform of the function below using MATLAB.

$$Y(s) = \frac{10}{s+2} + \frac{48}{(s+2)(s^2+16)}$$

Matlab:

```
clear all; close all
```

```
syms t s
```

```
Y = 10/(s+2) + 48/((s+2)*(s^2+16))
```

```
y = ilaplace(Y)
```

```
y=simplify(y)
```

Çözüm:

$$y = (62 \cdot \exp(-2 \cdot t))/5 - (12 \cdot \cos(4 \cdot t))/5 + (6 \cdot \sin(4 \cdot t))/5$$

Example

$$\mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 4} \right\} = \mathcal{L}^{-1} \left\{ \frac{3}{2} \cdot \frac{2}{s^2 + 2^2} \right\} = \frac{3}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 2^2} \right\} = \frac{3}{2} \sin 2t$$

$$\mathcal{L}^{-1} \left\{ \frac{2}{(s + 5)^4} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{3} \cdot \frac{6}{(s + 5)^4} \right\} = \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3!}{(s + 5)^4} \right\} = \frac{1}{3} e^{-5t} \mathcal{L}^{-1} \left\{ \frac{3!}{s^4} \right\} = \frac{1}{3} e^{-5t} t^3$$

$$\mathcal{L}^{-1} \left\{ \frac{s + 1}{s^2 + 4} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} + \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} = \cos 2t + \frac{1}{2} \sin 2t$$

Matlab:

```
clear all; close all
```

```
syms a s
```

```
f1 = 3/(s^2+4);
```

```
f2 = 2/(s+5)^4;
```

```
f3 = (s+1)/(s^2+4);
```

```
F1=ilaplace(f1)
```

```
F2=ilaplace(f2)
```

```
F3=ilaplace(f3)
```

Çözüm:

```
F1 = (3*sin(2*t))/2
```

```
F2 = (t^3*exp(-5*t))/3
```

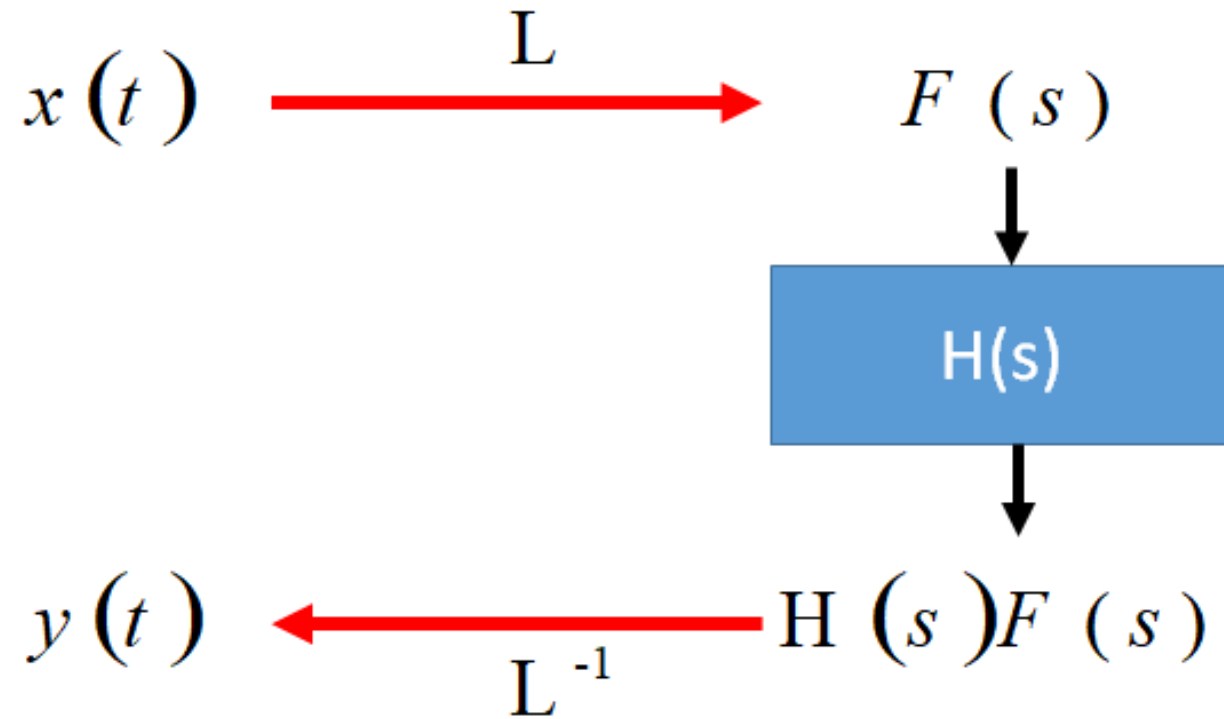
```
F3 = cos(2*t) + sin(2*t)/2
```

Transfer Function

Transfer Function

- Sistem bir girişe ve bir çıkışa sahipse, çıkışın girişin bir fonksiyonu olarak tanımlandığı durumlarda, onu s etki alanına (laplace kullanarak) dönüştürebilir ve sistemi bir transfer fonksiyonu ile tanımlanabilir.
- $X(s)$ 'in girdi olduğunu, $Y(s)$ 'nin çıkış olduğunu ve $H(s)$ 'nin transfer fonksiyonu olarak bilinen şey olduğunu varsayalım. Sistem çıkışı $Y(s) = H(s) * X(s)$ olarak tanımlanabilir.
- Transfer fonksiyonu $Y(s) / X(s)$ ile eşit olacaktır.
- Transfer fonksiyonunun pay ve paydası faktörize edilirse, bir kutup-sıfır tanımı yapılabilir. Sistemin yüksek frekansları söndürmek için yapılmış bir elektrik devresi ise, düşük geçiş filtresi, sadece kutup-sıfır çizimi filtrenin belirli frekanslara nasıl tepki verdiği söylenebilir. Bu, laplace dönüşümünün sadece küçük bir uygulamasıdır, ancak birçok mühendislik dalında kullanışlı olabilen transfer fonksiyonları ile lineer sistemlerin tanımlanmasını sağlar.

Transfer Function



Transfer fonksiyonu: Kazanç,
Kuvvetlendirme, Devre,
Süzgeç ...

Laplace transform can help us find $y(t)$ easily

Forms for Transfer Functions

$$F(s) = \frac{N(s)}{D(s)}$$

$$N(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

$$D(s) = b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0$$

$$F(s) = \frac{N(s)}{b_m (s - p_1)(s - p_2)\dots(s - p_m)}$$

The roots of $D(s)$ are called *poles* and they may be classified in four ways.

- 1. Real poles of first order.
- 2. Complex poles of first order (including purely imaginary poles)
- 3. Real poles of multiple order
- 4. Complex poles of multiple order (including purely imaginary poles)

Partial Fraction Expansion

Real Poles of First Order

$$F(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \dots + \frac{A_r}{s - p_r} + R(s)$$

$$A_k = \left. (s - p_k) F(s) \right]_{s=p_k}$$

$$f_1(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_r e^{p_r t}$$

Example. Determine inverse transform of function below.

$$F(s) = \frac{s+6}{s^2+3s+2} = \frac{s+6}{(s+1)(s+2)} \quad F(s) = \frac{s+6}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = (s+1)F(s) \Big|_{s=-1} = \frac{s+6}{s+2} \Big|_{s=-1} = \frac{-1+6}{-1+2} = 5$$

$$A_2 = (s+2)F(s) \Big|_{s=-2} = \frac{s+6}{s+1} \Big|_{s=-2} = \frac{-2+6}{-2+1} = -4$$

$$F(s) = \frac{5}{s+1} - \frac{4}{s+2} \quad f(t) = 5e^{-t} - 4e^{-2t}$$

Example. Determine exponential portion of inverse transform of function below.

$$F(s) = \frac{50(s+3)}{(s+1)(s+2)(s^2+2s+5)} \quad F_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = \left. \frac{50(s+3)}{(s+2)(s^2+2s+5)} \right]_{s=-1} = \frac{(50)(2)}{(1)(4)} = 25$$

$$A_2 = \left. \frac{50(s+3)}{(s+1)(s^2+2s+5)} \right]_{s=-2} = \frac{(50)(1)}{(-1)(5)} = -10$$

$$f_1(t) = 25e^{-t} - 10e^{-2t}$$

Partial Fraction Expansion for First-Order Complex Poles

$$(s^2 + bs + c) \Rightarrow p_{1,2} = -\alpha \pm i\omega$$

$$F(s) = \frac{As + B}{s^2 + bs + c} + R(s)$$

Example. Complete the inverse transform.

$$\frac{50(s+3)}{(s+1)(s+2)(s^2+2s+5)} = \frac{25}{s+1} - \frac{10}{s+2} + \frac{As+B}{s^2+2s+5}$$

$$\frac{50(3)}{(1)(2)(5)} = \frac{25}{1} - \frac{10}{2} + \frac{B}{5} \quad B = -25$$

$$\frac{50(4)}{(2)(3)(8)} = \frac{25}{2} - \frac{10}{3} + \frac{A+B}{8} \quad A = -15$$

$$F(s) = \frac{25}{s+1} - \frac{10}{s+2} + \frac{-15s-25}{s^2+2s+5}$$

Example. Continuation.

$$F_2(s) = \frac{-15s - 25}{s^2 + 2s + 5}$$

$$s^2 + 2s + 5 = s^2 + 2s + 1 + 5 - 1 = (s + 1)^2 + (2)^2$$

$$F_2(s) = \frac{-15s - 25}{(s + 1)^2 + (2)^2} = \frac{-15(s + 1)}{(s + 1)^2 + (2)^2} + \frac{-5(2)}{(s + 1)^2 + (2)^2}$$

$$f(t) = f_1(t) + f_2(t)$$

$$= 25e^{-t} - 10e^{-2t} - 15e^{-t} \cos 2t - 5e^{-t} \sin 2t$$

Second-Order Real Poles

Assume that $F(s)$ contains a denominator factor of the form $(s+\alpha)^2$. The expansion will take the form shown below.

$$F(s) = \frac{C_1}{(s + \alpha)^2} + \frac{C_2}{s + \alpha} + R(s)$$

$$C_1 = (s + \alpha)^2 F(s) \Big|_{s=-\alpha}$$

$$f_1(t) = C_1 t e^{-\alpha t} + C_2 e^{-\alpha t} = (C_1 t + C_2) e^{-\alpha t}$$

Example. Determine inverse transform of function below.

$$F(s) = \frac{60}{s(s+2)^2}$$

$$F(s) = \frac{60}{s(s+2)^2} = \frac{A}{s} + \frac{C_1}{(s+2)^2} + \frac{C_2}{(s+2)}$$

$$A = sF(s) \Big|_{s=0} = \frac{60}{(s+2)^2} \Big|_{s=0} = \frac{60}{(0+2)^2} = 15$$

$$C_1 = (s+2)^2 F(s) \Big|_{s=-2} = \frac{60}{s} \Big|_{s=-2} = \frac{60}{-2} = -30$$

Example. Continuation.

$$F(s) = \frac{60}{s(s+2)^2} = \frac{15}{s} - \frac{30}{(s+2)^2} + \frac{C_2}{s+2}$$

$$\frac{60}{(1)(1+2)^2} = \frac{15}{1} - \frac{30}{(1+2)^2} + \frac{C_2}{(1+2)} \quad C_2 = -15$$

$$F(s) = \frac{60}{s(s+2)^2} = \frac{15}{s} - \frac{30}{(s+2)^2} - \frac{15}{s+2}$$

$$f(t) = 15 - 30te^{-2t} - 15e^{-2t} = 15 - 15e^{-2t}(1+2t)$$

Differentiation & Integration

Laplace Transform Operations

$f(t)$	$F(s)$	
$f'(t)$	$sF(s) - f(0)$	O-1
$\int_0^t f(t)dt$	$\frac{F(s)}{s}$	O-2
$e^{-\alpha t} f(t)$	$F(s + \alpha)$	O-3
$f(t - T)u(t - T)$	$e^{-sT} F(s)$	O-4
$f(0)$	$\lim_{s \rightarrow \infty} sF(s)$	O-5
$\lim_{t \rightarrow \infty} f(t)$	$\lim_{s \rightarrow 0} sF(s)^*$	O-6

*Poles of $sF(s)$ must have negative real parts.

Transforms of Derivatives and Integrals. ODEs

Theorem

Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$, Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s), \quad \text{thus} \quad \int_0^t f(\tau) d\tau = \mathcal{L}^{-1} \left\{ \frac{1}{s} F(s) \right\}.$$

Differentiation

$$\mathbf{L}[f(t)] = F(s)$$

$$\mathbf{L}[f'(t)] = sF(s) - f(0^-)$$

$$\begin{aligned}\mathbf{L}[f''(t)] &= s(sF(s) - f(0^-)) - f'(0^-) \\ &= s^2 F(s) - sf(0^-) - f'(0^-)\end{aligned}$$

⋮

$$\mathbf{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) \cdots - f^{(n-1)}(0^-)$$

Significant Operations for Solving Differential Equations

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Procedure for Solving DEs

$$b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = f(t) \quad L \left[b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y \right] = L[f(t)]$$

$$b_2 \left[s^2 Y(s) - sy(0) - y'(0) \right] \\ + b_1 \left[sY(s) - y(0) \right] + b_0 Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{b_2 s^2 + b_1 s + b_0} + \frac{sb_2 y(0) + b_2 y'(0) + b_1 y(0)}{b_2 s^2 + b_1 s + b_0}$$

Solution of ODE

Significant Operations for Solving Differential Equations

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

$f(0)$ and $f'(0)$, initial condition
at $t = 0$

Laplace Transform for ODEs

- Equation with initial conditions

$$\frac{d^2 y}{dt^2} + y = 1, \quad y(0) = y'(0) = 0$$

- Laplace transform is linear

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1)$$

- Apply derivative formula

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s}$$

- Rearrange

$$\mathcal{L}(y) = \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}$$

- Take the inverse

$$y = 1 - \cos t$$

Example

$$\frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

$$s^2 Y(s) + 6s Y(s) + 8Y(s) = 2/s$$

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

- ODE w/initial conditions
- Apply Laplace transform to each term
- Solve for Y(s)
- Apply partial fraction expansion
- Apply inverse Laplace transform to each term

Example

$$y'' + 3y' + 2y = e^{-t} \text{ given that } y(0) = 4, y'(0) = 5$$

$$L\left(\frac{dy}{dt}\right) = L(y') = sY(s) - y(0)$$

$$L\left(\frac{d^2y}{dt^2}\right) = L(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$L\left(\frac{d^3y}{dt^3}\right) = L(y''') = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Taking Laplace transform on both sides & Let $L(y(t)) = Y(s)$.

$$L(y'') + 3L(y') + 2L(y) = L(e^{-t})$$

Using formulae

$$\{s^2Y(s) - sy(0) - y'(0)\} + 3\{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s+1}$$

Substituting given conditions, $y(0) = 4, y'(0) = 5$

We have

$$(s^2 - 3s + 2)Y(s) - 4s - 5 - 12 = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s^2+3s+2)} \left(4s + 17 + \frac{1}{s+1}\right)$$

$$Y(s) = \frac{4s^2+21s+18}{(s^2+3s+2)(s+1)}$$

Using Partial Fractions, We get

$$Y(s) = \frac{-8}{s+2} + \frac{12}{s+1} + \frac{1}{(s+1)^2}$$

Taking inverse Laplace transforms,

$$L^{-1}(Y(s)) = -8L^{-1}\left(\frac{1}{s+2}\right) + 12L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$y(t) = 12e^{-t} - 8e^{-2t} + te^{-t}.$$

(Laplace Transform for ODE) To solve this differential equation

$$y'' - 6y' + 8y = 0, \quad y(0) = -1, \quad y'(0) = 2$$

```
clear all; close all
```

```
syms t s Y
```

```
d='D(D(y))(t)-6*D(y)(t)+8*y(t)=0'
```

```
Ld=laplace(d,t,s)
```

```
Le=subs(Ld,{'laplace(y(t), t, s)', 'y(0)', 'D(y)(0)'}, {Y,-1,2})
```

```
Y=solve(Le,Y)
```

```
y=ilaplace(Y,s,t)
```

$$d = D(D(y))(t) - 6 * D(y)(t) + 8 * y(t) = 0$$

$$Ld = 6 * y(0) - D(y)(0) - 6 * s * \text{laplace}(y(t), t, s) - s * y(0) + s^2 * \text{laplace}(y(t), t, s) + 8 * \text{laplace}(y(t), t, s) == 0$$

$$Le = 8 * Y + s - 6 * Y * s + Y * s^2 - 8 == 0$$

$$Y = -(s - 8) / (s^2 - 6 * s + 8)$$

$$y = 2 * \exp(4 * t) - 3 * \exp(2 * t)$$

Example 6 (Resonance Phenomenon) To solve this differential equation

$$y'' + 9y = 6\cos(3t), \quad y(0) = 0, y'(0) = 0$$

```
clear all; close all
```

```
syms t s Y
```

```
d='D(D(y))(t)+9*y(t)=6*cos(3*t)'
```

```
Ld=laplace(d,t,s)
```

```
Le=subs(Ld,{'laplace(y(t), t, s)', 'y(0)', 'D(y)(0)'}, {Y,0,0})
```

```
Y=solve(Le,Y)
```

```
y=ilaplace(Y,s,t)
```

```
ezplot(y,[0,30])
```

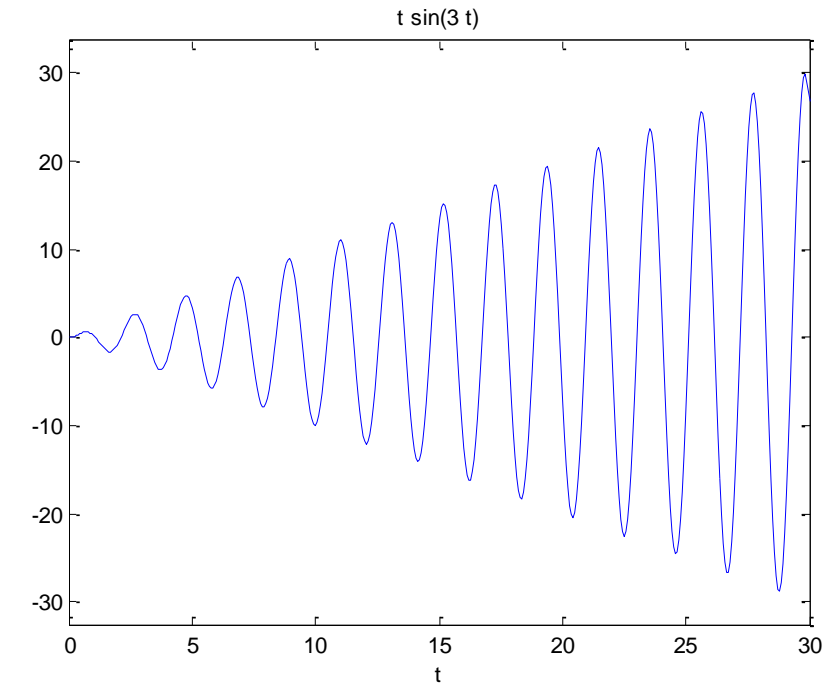
```
d = D(D(y))(t)+9*y(t)=6*cos(3*t)
```

```
Ld = s^2*laplace(y(t), t, s) - s*y(0) - D(y)(0) + 9*laplace(y(t), t, s) == (6*s)/(s^2 + 9)
```

```
Le = Y*s^2 + 9*Y == (6*s)/(s^2 + 9)
```

```
Y =(6*s)/(s^2 + 9)^2
```

```
y = t*sin(3*t)
```



Example 8 (Impulse Forcing) To solve this differential equation with impulse forcing

$$y'' + 2y' + y = 3\delta(t - 2), \quad y(0) = 2, y'(0) = 1$$

```
clear all; close all
```

```
syms t s Y
```

```
d='D(D(y))(t)+2*D(y)(t)+y(t)=3*dirac(t-2)'
```

```
Ld=laplace(d,t,s)
```

```
Le=subs(Ld,{'laplace(y(t), t, s)', 'y(0)', 'D(y)(0)'}, {Y,2,1})
```

```
Y=solve(Le,Y)
```

```
y=ilaplace(Y,s,t)
```

```
d =D(D(y))(t)+2*D(y)(t)+y(t)=3*dirac(t-2)
```

```
Ld =2*s*laplace(y(t), t, s) - D(y)(0) - 2*y(0) - s*y(0) + s^2*laplace(y(t), t, s) + laplace(y(t), t, s) == 3*exp(-2*s)
```

```
Le =Y - 2*s + 2*Y*s + Y*s^2 - 5 == 3*exp(-2*s)
```

```
Y =(2*s + 3*exp(-2*s) + 5)/(s^2 + 2*s + 1)
```

```
y =2*exp(-t) + 3*t*exp(-t) + 3*heaviside(t - 2)*exp(2 - t)*(t - 2)
```

Not: $u(t-2)=\text{heaviside}(t - 2)$

Örnek: $y'' + y = \cos(2t)$, $y(0) = 1$, $y'(0) = 0$

```
clear all; close all
```

```
syms y(t) t Y s
```

```
Dy = diff(y,t);
```

```
D2y = diff(Dy,t);
```

```
y = D2y + y == cos(2*t)
```

```
Ld = laplace(y)
```

```
Le = subs(Ld,{'laplace(y(t), t, s)', 'y(0)', 'D(y)(0)'}, {Y,1,0})
```

```
Y = solve(Le,Y)
```

```
y = ilaplace(Y,s,t)
```

```
y(t) = D(D(y))(t) + y(t) == cos(2*t)
```

```
Ld(s) = s^2*laplace(y(t), t, s) - s*y(0) - D(y)(0) + laplace(y(t), t, s) == s/(s^2 + 4)
```

```
Le(s) = Y*s^2 - s + Y == s/(s^2 + 4)
```

```
Y = (s + s/(s^2 + 4))/(s^2 + 1)
```

```
y = (4*cos(t))/3 - cos(2*t)/3
```

Example. Solve DE shown below.

$$\frac{dy}{dt} + 2y = 12 \quad y(0) = 10$$

$$L\left[\frac{dy}{dt}\right] + 2L[y] = L[12]$$

$$sY(s) - 10 + 2Y(s) = \frac{12}{s}$$

$$(s + 2)Y(s) = 10 + \frac{12}{s}$$

$$Y(s) = \frac{10}{s + 2} + \frac{12}{s(s + 2)}$$

$$\frac{12}{s(s + 2)} = \frac{A_1}{s} + \frac{A_2}{s + 2}$$

$$A_1 = s \left[\frac{12}{s(s + 2)} \right]_{s=0} = \left[\frac{12}{s + 2} \right]_{s=0} = 6$$

$$A_2 = (s + 2) \left[\frac{12}{s(s + 2)} \right]_{s=-2} = \left[\frac{12}{s} \right]_{s=-2} = -6$$

$$Y(s) = \frac{10}{s + 2} + \frac{6}{s} - \frac{6}{s + 2} = \frac{6}{s} + \frac{4}{s + 2}$$

$$y(t) = 6 + 4e^{-2t}$$

Example. Solve DE shown below.

$$\frac{dy}{dt} + 2y = 12 \sin 4t \quad y(0) = 10$$

$$sY(s) - 10 + 2Y(s) = \frac{12(4)}{s^2 + 16} \quad Y(s) = \frac{10}{s + 2} + \frac{48}{(s + 2)(s^2 + 16)}$$

$$\frac{48}{(s + 2)(s^2 + 16)} = \frac{A}{s + 2} + \frac{B_1s + B_2}{s^2 + 16} \quad A = \left. \frac{48}{s^2 + 16} \right]_{s=-2} = \frac{48}{20} = 2.4$$

$$\frac{48}{(s + 2)(s^2 + 16)} = \frac{2.4}{s + 2} + \frac{B_1s + B_2}{s^2 + 16} \quad \frac{48}{(2)(16)} = \frac{2.4}{2} + \frac{B_2}{16} \quad B_2 = 4.8$$

$$\frac{48}{(1)(17)} = \frac{2.4}{1} + \frac{-B_1 + B_2}{17} \quad B_1 = -2.4$$

Example. Continuation.

$$Y(s) = \frac{10}{s+2} + \frac{2.4}{s+2} - \frac{2.4s}{s^2+16} + \frac{4.8}{s^2+16}$$

$$y(t) = 12.4e^{-2t} - 2.4\cos 4t + 1.2\sin 4t$$

Example. Solve DE shown below.

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 24 \quad y(0) = 10 \text{ and } y'(0) = 0$$

$$s^2 Y(s) - 10s - 0 + 3[sY(s) - 10] + 2Y(s) = \frac{24}{s}$$

$$\begin{aligned} Y(s) &= \frac{24}{s(s^2 + 3s + 2)} + \frac{10s + 30}{s^2 + 3s + 2} \\ &= \frac{24}{s(s+1)(s+2)} + \frac{10s + 30}{(s+1)(s+2)} \end{aligned}$$

Example. Continuation.

$$\frac{24}{s(s+1)(s+2)} = \frac{12}{s} - \frac{24}{s+1} + \frac{12}{s+2}$$

$$\frac{10s+30}{(s+1)(s+2)} = \frac{20}{s+1} - \frac{10}{s+2}$$

$$F(s) = \frac{12}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$f(t) = 12 - 4e^{-t} + 2e^{-2t}$$

Example. Solve DE shown below.

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = 20 \quad y(0) = 0 \text{ and } y'(0) = 10$$

$$s^2 Y(s) - 0 - 10 + 2[sY(s) - 0] + 5Y(s) = \frac{20}{s}$$

$$Y(s) = \frac{20}{s(s^2 + 2s + 5)} + \frac{10}{s^2 + 2s + 5}$$

$$\frac{20}{s(s^2 + 2s + 5)} = \frac{4}{s} + \frac{As + B}{s^2 + 2s + 5}$$

$$\frac{20}{(1)(1 + 2 + 5)} = \frac{4}{1} + \frac{A + B}{(1 + 2 + 5)}$$

$$\frac{20}{(-1)(1 - 2 + 5)} = \frac{4}{-1} + \frac{-A + B}{(1 - 2 + 5)}$$

$$A = -4 \quad B = -8$$

Example. Continuation.

$$Y(s) = \frac{4}{s} + \frac{-4s - 8}{s^2 + 2s + 5} + \frac{10}{s^2 + 2s + 5} = \frac{4}{s} + \frac{-4s + 2}{s^2 + 2s + 5}$$

$$s^2 + 2s + 5 = s^2 + 2s + 1 + 5 - 1 = (s + 1)^2 + (2)^2$$

$$Y(s) = \frac{4}{s} + \frac{-4(s + 1)}{(s + 1)^2 + (2)^2} + \frac{3(2)}{(s + 1)^2 + (2)^2}$$

$$y(t) = 4 - 4e^{-t} \cos 2t + 3e^{-t} \sin 2t$$

İlk(0) ve Son Değer(∞) Teoremleri **Initial- and Final-Value Theorems**

The Laplace Transform

Theorem: Initial Value

If the function $f(t)$ and its first derivative are Laplace transformable and $f(t)$ has the Laplace transform $F(s)$, and the $\lim_{s \rightarrow \infty} sF(s)$ exists, then

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{t \rightarrow 0} f(t) = f(0)$$

*Initial Value
Theorem*

The utility of this theorem lies in not having to take the inverse of $F(s)$ in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

*Final-value
Theorem*

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

The Laplace Transform

Example: Initial Value

Given;

$$F(s) = \frac{(s + 2)}{(s + 1)^2 + 5^2}$$

Find $f(0)$

$$\begin{aligned} f(0) &= \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} s \frac{(s + 2)}{(s + 1)^2 + 5^2} = \lim_{s \rightarrow \infty} \left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25} \right] \\ &= \lim_{s \rightarrow \infty} \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1 \end{aligned}$$

Kararlılık

- Başlangıç ve final değerleri sayısal ise yani sonsuza gitmiyorsa sistem ya da sinyal kararlıdır.
- Laplace dönüşümünde $f(t)$ fonksiyonunun Laplace dönüşümü $F(s)$ ise $f(t)$ fonksiyonun kararsız bölgesi $F(s)$ fonksiyonun paydasını sıfır yapan değerlerdir.

The Laplace Transform

Example: Final Value

Given:

$$F(s) = \frac{(s+2)^2 - 3^2}{(s+2)^2 + 3^2}$$

note $F^{-1}(s) = te^{-2t} \cos 3t$

Find $f(\infty)$.

$$f(\infty) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} s \frac{(s+2)^2 - 3^2}{(s+2)^2 + 3^2} = \mathbf{0}$$

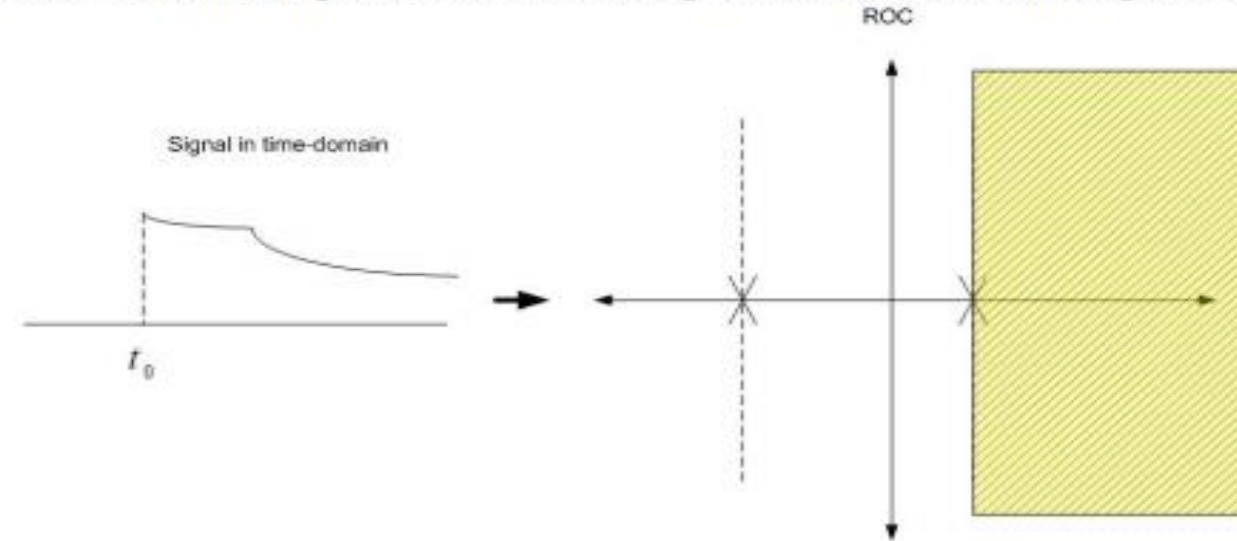
Yakınsama Bölgesinin Özellikleri

- Kompleks değişken $s = \sigma + j\omega$ 'nın imajiner kısmı, $\omega = \text{Im} [s]$ yakınsama açısından bir etkisi olmadığından, ROC sadece gerçek $\sigma = \text{Re} [s]$ kısmı tarafından belirlenir.
- ROC $j\omega$ eksenine paraleldir.
- $x(t)$ sonlu bir sinyal ise (integrali sonlu aralıkta ise) ROC bütün s -düzlemidir.
- $x(t)$ sağ taraflı ise (sol tarafı sıfır ise) ROC'da sağ taraflıdır.
- $x(t)$ sol taraflı ise (sağ tarafı sıfır ise) ROC'da sol taraflıdır.

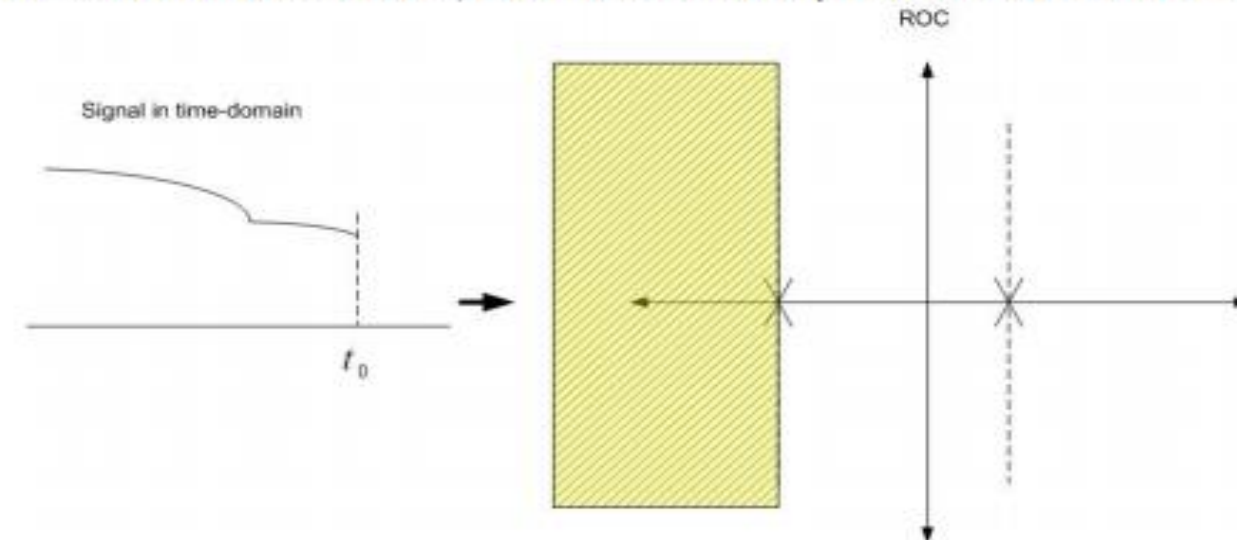
Five Simple Rules to Find Region of Convergence (ROC)

1- ROC must be bounded by poles or extends to infinity (it means ROC can not include poles).

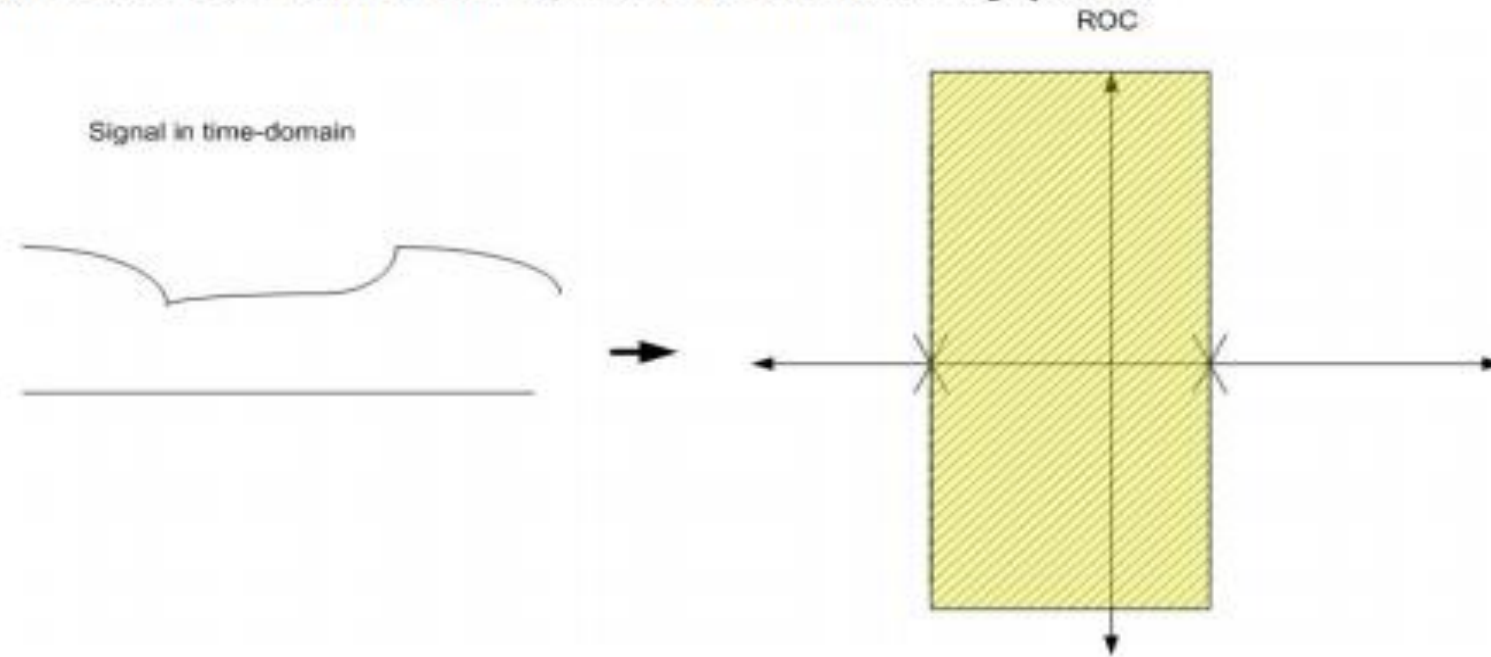
2- If the signal in time-domain is right-sided, ROC is right-sided (ROC is the right side of rightmost pole).



3- If the signal in time-domain is left-sided, ROC is left-sided (ROC is the left side of leftmost pole).



4- If the signal in time-domain is two-sided, ROC is bounded by poles.



5- If the signal in time-domain has a finite duration, ROC is everywhere.

Yakınsaklık Bölgesi

- Görüldüğü gibi Laplace dönüşümü, karmaşık s -düzleminde $j\omega$ -ekseni üzerinde hesaplandığında sürekli-zaman Fourier dönüşümünü verir. !!!

$$X(s)|_{s=j\omega} = F\{x(t)\}$$

- $x(t)e^{-\sigma t}$ işaretinin Fourier dönüşümü de $x(t)$ işaretinin Laplace dönüşümünü verir.

- Bu durumda:

1-) Bir $x(t)$ işaretinin Laplace dönüşümünün var olabilmesi için $x(t)e^{-\sigma t}$ işaretinin Fourier dönüşümü yakınsamalıdır. Verilen bir $x(t)$ işareti için, Laplace dönüşümünün var olduğu s değerleri kümesine **YAKINSAKLIK BÖLGESİ (Region of Converge, ROC)** denir.

2-) Eğer ROC imajiner eksen ($s=j\omega$) içeriyorsa, işaretin Fourier dönüşümü de vardır.

3-) Bazı işaretler için Fourier dönüşümü yakınsamaz iken Laplace dönüşümü yakınsayabilir.

Laplace Dönüşümü ve Yakınsaklık Bölgesi

Örnekler incelediğinde farklı iki işarete ait Laplace dönüşümlerinin cebirsel olarak birbirine eşit olduğu görülür.

$$-e^{-at}u(-t) \xrightarrow{L} X(s) = \frac{1}{s+a}, \quad \text{Re}\{s\} < -a$$

$$e^{-at}u(t) \xrightarrow{L} X(s) = \frac{1}{s+a}, \quad \text{Re}\{s\} > -a$$

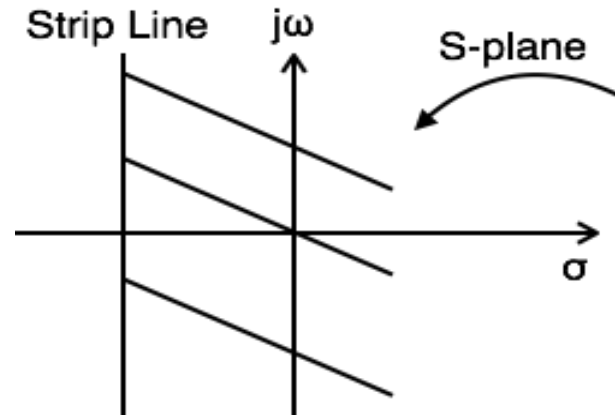
S=-a belirsilik ifade etmektedir.

Fakat eşitliklerin geçerli olduğu yakınsaklık bölgesinin $\{\text{Re}\{s\} < -a\}$ ve $\{\text{Re}\{s\} > -a\}$ birbirinden farklı olduğuna dikkat ediniz.

Bu durumda Laplace dönüşümü için cebirsel ifadenin yanısıra yakınsaklık bölgeside belirtilmelidir.

Properties of ROC of Laplace Transform

- ROC contains strip lines parallel to $j\omega$ axis in s-plane.
 - If $x(t)$ is absolutely integral and it is of finite duration, then ROC is entire s-plane.
 - If $x(t)$ is a right sided sequence then ROC : $\text{Re}\{s\} > \sigma_0$.
 - If $x(t)$ is a left sided sequence then ROC : $\text{Re}\{s\} < \sigma_0$.
 - If $x(t)$ is a two sided sequence then ROC is the combination of two regions.
- ROC can be explained by making use of examples given below:

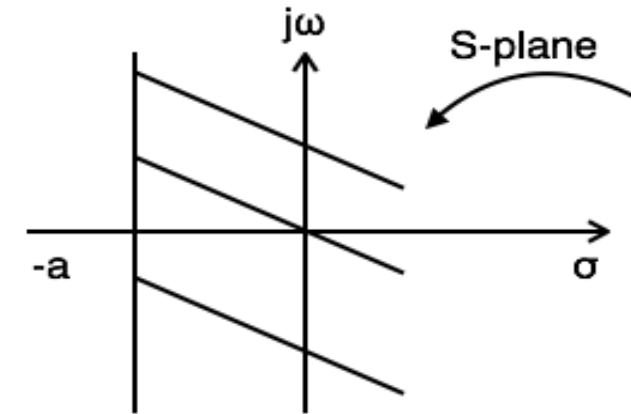


Example 1: Find the Laplace transform and ROC of $x(t) = e^{-at} u(t)$

$$L.T[x(t)] = L.T[e^{-at} u(t)] = \frac{1}{s+a}$$

$$\text{Re} > -a$$

$$\text{ROC : } \text{Re} s > -a$$

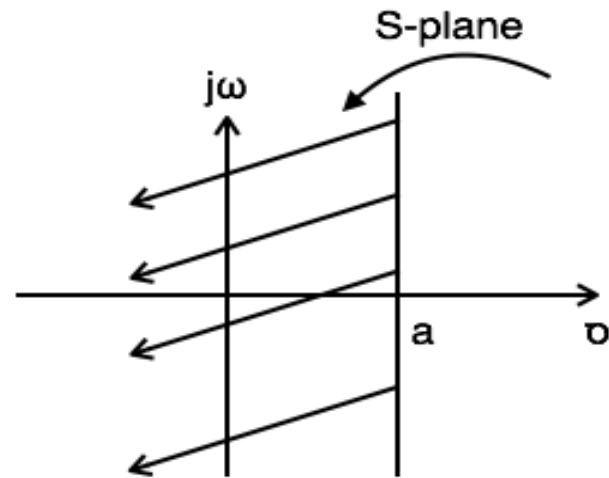


Example 2: Find the Laplace transform and ROC of $x(t) = e^{at}u(-t)$

$$L.T[x(t)] = L.T[e^{at}u(t)] = \frac{1}{s-a}$$

$$Res < a$$

$$ROC : Res < a$$

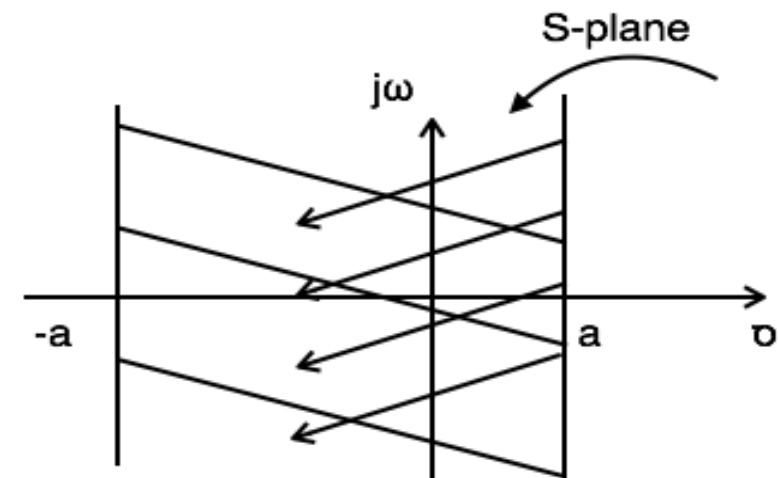


Example 3: Find the Laplace transform and ROC of $x(t) = e^{-at}u(t) + e^{at}u(-t)$

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{s+a} + \frac{1}{s-a}$$

$$\text{For } \frac{1}{s+a} \text{ Re}\{s\} > -a$$

$$\text{For } \frac{1}{s-a} \text{ Re}\{s\} < a$$

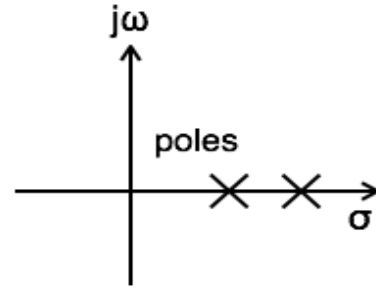


Referring to the above diagram, combination region lies from $-a$ to a . Hence,

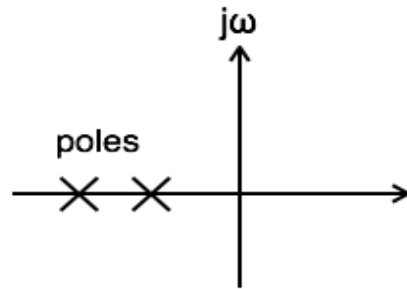
$$ROC : -a < Res < a$$

Causality and Stability

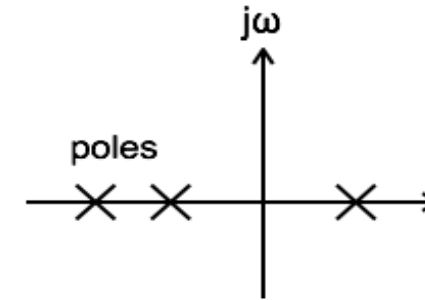
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



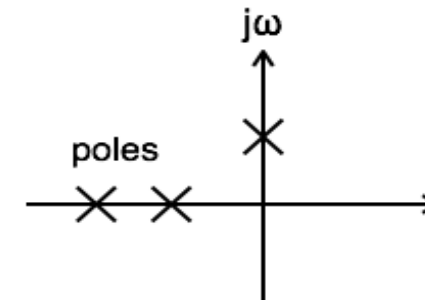
- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transfer function lies on the $j\omega$ axis of s-plane.



Nedensellik – ROC

a) Causality:

Eğer bir sistemin çıkışı, şu anki girişi dâhil olmak üzere önceki değerlerine bağlı ise sisteme nedensel (causal) sistem denir. Tüm gerçek zamanlı fiziksel sistemler nedenseldir; çünkü zaman sadece ileriye akar.

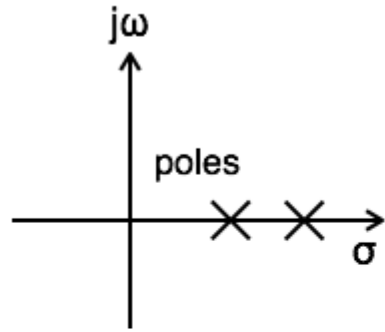
Sistem nedensel ise, ROC sağ taraflı olmalıdır (diğer yol mutlaka doğru değildir). Nedensellik Bulmak İçin Üç Basit Kural:

- 1- ROC sağ yönlü değilse sistem nedensel değildir.
- 2- ROC sağ yönlü ve Laplace dönüşümü rasyonel bir fonksiyon ise sistem nedenseldir.
- 3- ROC sağ taraflıysa ve Laplace dönüşümü rasyonel bir fonksiyon değilse, sistemi zaman alanında bulmalıyız ve sistemin nedensel olup olmadığına karar verebiliriz.

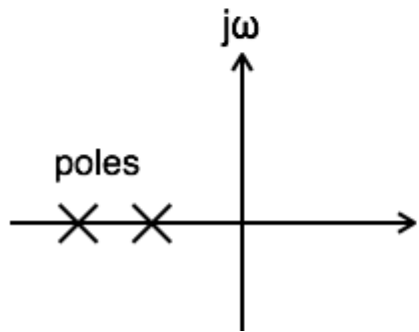
b) Kararlılık: Sistem, ancak ve ancak ROC kompleks eksen içeriyorsa kararlıdır.

Causality and Stability

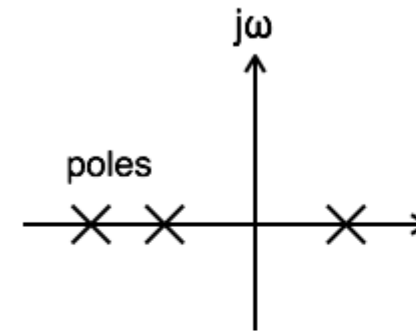
- For a system to be causal, all poles of its transfer function must be right half of s-plane.



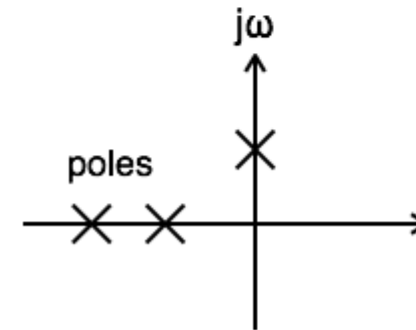
- A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



- A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



- A system is said to be marginally stable when at least one pole of its transfer function lies on the $j\omega$ axis of s-plane.



ROC of Basic Functions (Kararlılık bölgesi)

$f(t)$	$F(s)$	ROC
$u(t)$	$\frac{1}{s}$	ROC: $\text{Re}\{s\} > 0$
$t u(t)$	$\frac{1}{s^2}$	ROC: $\text{Re}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	ROC: $\text{Re}\{s\} > 0$
$e^{at} u(t)$	$\frac{1}{s - a}$	ROC: $\text{Re}\{s\} > a$

$e^{-at} u(t)$	$\frac{1}{s + a}$	ROC: $\text{Re}\{s\} > -a$
$e^{at} u(t)$	$-\frac{1}{s - a}$	ROC: $\text{Re}\{s\} < a$
$e^{-at} u(-t)$	$-\frac{1}{s + a}$	ROC: $\text{Re}\{s\} < -a$
$t e^{at} u(t)$	$\frac{1}{(s - a)^2}$	ROC: $\text{Re}\{s\} > a$

ROC of Basic Functions

$$t e^{-at} u(-t)$$

$$-\frac{1}{(s+a)^2}$$

$$\text{ROC: } \text{Re}\{s\} < -a$$

$$t^n e^{-at} u(-t)$$

$$-\frac{n!}{(s+a)^{n+1}}$$

$$\text{ROC: } \text{Re}\{s\} < -a$$

$$e^{-at} \cos bt$$

$$\frac{s+a}{(s+a)^2 + b^2}$$

$$e^{-at} \sin bt$$

$$\frac{b}{(s+a)^2 + b^2}$$

ROC

Find the ROC of $x(t) = e^{-2t} u(t) + e^{-3t} u(t)$.

- a) $\sigma > 2$
- b) $\sigma > 3$
- c) $\sigma > -3$
- d) $\sigma > -2$

Answer: d

Explanation: Given $x(t) = e^{-2t} u(t) + e^{-3t} u(t)$

Laplace transform, $L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$

$$X(s) = 1/(s+2) + 1/(s+3)$$

ROC is $\{\sigma > -2\} \cap \{\sigma > -3\}$

Hence, the ROC is $\sigma > -2$.

Laplace dönüşümü $1/(s+2) + 1/(s+3)$ olan ifadenin ROC katsayısı nedir? (ROC'u bulmak için Laplace dönüşümünün kökleri bulunur, ROC, pozitif yöne doğru paydanın köklerin kümesel kesişim değeridir.

Laplace dönüşümü $1/(s-2) + 1/(s-3)$ olan ifadenin ROC katsayısı nedir? (ROC'u bulmak için Laplace dönüşümünün kökleri bulunur, ROC, pozitif yöne doğru paydanın köklerin kümesel kesişim değeridir. $\sigma > 3$

Region of Convergence (Yakınsama Bölgesi - ROC)

$$f(t) = 1$$

$$F(s) = \frac{1}{s}$$

Dikkat: Bir fonksiyonun yakınsaması için değerinin sonsuzdan farklı olması gerekir.

$\sigma \leq 0$ olduğu durumlar kararsız olur.

If $\text{Re}[s]=\sigma > 0$, yakınsar

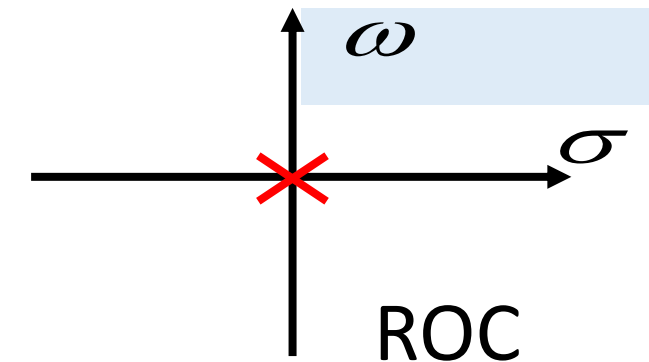
Time domain: 1



s-domain: $1/s$

Time domain: $u(t)$
(Only consider $t > 0$)

Yakınsaması
için ROC: $\sigma > 0$



ROC

The Laplace transform of $x(t) = e^{-at}u(t)$ is:

$$\begin{aligned} X(s) &= \mathcal{L}[x(t)] = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-at} e^{-(\sigma+j\omega)t} dt \\ &= \frac{1}{a + \sigma + j\omega} e^{-(a+\sigma+j\omega)t} \Big|_0^{\infty} \end{aligned}$$

For this integral to converge, we need to have

$$a + \sigma > 0 \quad \text{or} \quad \sigma = \text{Re}[s] > -a$$

and the Laplace transform is

$$X(s) = \frac{1}{(\sigma + a) + j\omega} = \frac{1}{s + a}$$

As a special case where $a = 0$, $x(t) = u(t)$ and we have

$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \sigma = \text{Re}[s] > 0$$

ÖRNEK : $x(t) = e^{-at}u(t)$ işaretinin Laplace dönüşümünü hesaplayınız.

ÇÖZÜM: Bu işaret için Fourier dönüşümü önceki haftalarda aşağıdaki gibi hesaplanmıştır.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a + j\omega}, \quad a > 0$$

İşaretin Laplace dönüşümü ise,

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-s t} dt = \int_0^{\infty} e^{-at} e^{-s t} dt = \int_0^{\infty} e^{-(s+a)t} dt$$

$$X(s) \Big|_{s=\sigma+j\omega} = \int_0^{\infty} e^{-(\sigma+a)t} e^{-j\omega t} dt = \frac{1}{(\sigma+a) + j\omega}, \quad \sigma+a > 0$$

veya, $s = \sigma + j\omega \rightarrow X(s) = \frac{1}{s+a}, \quad \text{Re}\{s\} > -a$

Example 3:

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

The Laplace transform is linear, and $X(s)$ is the sum of the transforms for the two terms:

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}, \quad (\sigma > -a), \quad \mathcal{L}[e^{at}u(-t)] = \frac{-1}{s-a}, \quad (\sigma < a)$$

If $a > 0$, i.e., $x(t)$ decays when $|t| \rightarrow \infty$, the intersection of the two ROCs is $-a < \sigma < a$, and we have:

$$\mathcal{L}[x(t)] = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2 - a^2}$$

However, if $a < 0$, i.e., $x(t)$ grows without a bound when $|t| \rightarrow \infty$, the intersection of the two ROCs is an empty set, the Laplace transform does not exist.

Örnek: Yakınsama Bölgesi - ROC

ÖRNEK: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$ işaretinin Laplace dönüşümünü hesaplayınız..

$$X(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)] e^{-s t} dt = 3 \int_0^{\infty} e^{-2t} e^{-s t} dt - 2 \int_0^{\infty} e^{-t} e^{-s t} dt$$

$$e^{-2t}u(t) \xrightarrow{L} X(s) = \frac{1}{s+2}, \quad \text{Re}\{s\} > -2$$

$$e^{-t}u(t) \xrightarrow{L} X(s) = \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2} \quad \text{Re}\{s\} > -1$$

her iki koşulun sağlandığı bölge...

- Herbir fonksiyonun Laplace dönüşümü alınır.
- Laplace dönüşümü alınan herbir fonksiyonun paydasını sıfır yapan değerler bulunur. (Fonksiyon sonsuz gider; yani kararsızlık vardır.)
- En büyük değerden sağa doğru olan bölge kararlıdır. Her iki koşulun sağlandığı bölge.
- Kompleks değer çıkarsa reel kısmı alınır.
- Kökler biri büyük diğer küçük ise aralık alınır.

ÖRNEK: $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$ işaretinin Laplace dönüşümünü hesaplayınız..

$$x(t) = \left[e^{-2t} + e^{-t} \left(\frac{e^{3jt} + e^{-3jt}}{2} \right) \right] u(t) = \left[e^{-2t} + \frac{1}{2} e^{-(1-3j)t} + \frac{1}{2} e^{-(1+3j)t} \right] u(t)$$

$$e^{-2t}u(t) \xrightarrow{L} X(s) = \frac{1}{s+2}, \quad \text{Re}\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \xrightarrow{L} X(s) = \frac{1}{s+(1-3j)}, \quad \text{Re}\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \xrightarrow{L} X(s) = \frac{1}{s+(1+3j)}, \quad \text{Re}\{s\} > -1$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2} \frac{1}{s+(1+3j)} + \frac{1}{2} \frac{1}{s+(1-3j)} = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s+2)}, \quad \text{Re}\{s\} > -1$$

Determine the Laplace transforms (including the regions of convergence) of each of the following signals:

a. $x_1(t) = e^{-2(t-3)}u(t-3)$

$X_1 =$

$$\frac{e^{-3s}}{s+2}$$

ROC:

$$\operatorname{Re}(s) > -2$$

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} x_1(t)e^{-st} dt = \int_{-\infty}^{\infty} e^{-2(t-3)}u(t-3)e^{-st} dt = e^6 \int_3^{\infty} e^{-(s+2)t} dt \\ &= e^6 \left. \frac{e^{-(s+2)t}}{-(s+2)} \right|_3^{\infty} = \frac{e^{-3s}}{s+2}; \quad \operatorname{Re}(s) > -2 \end{aligned}$$

$$x_2(t) = (1 - (1 - t)e^{-3t})u(t)$$

$X_2 =$

$$\frac{4s + 9}{s(s + 3)^2}$$

ROC:

$$\text{Re}(s) > 0$$

Treat this as the sum of 3 signals: $x_2(t) = x_{2a}(t) + x_{2b}(t) + x_{2c}(t)$, where $x_{2a}(t) = u(t)$, $x_{2b}(t) = -e^{-3t}u(t)$, and $x_{2c}(t) = te^{-3t}u(t)$.

$$X_{2a}(s) = \int_0^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{1}{s}; \quad \text{Re}(s) > 0$$

$$X_{2b}(s) = \int_0^{\infty} -e^{-3t}e^{-st} dt = \frac{-e^{-(s+3)t}}{-(s+3)} \Big|_0^{\infty} = -\frac{1}{s+3}; \quad \text{Re}(s) > -3$$

$$X_{2c}(s) = \int_0^{\infty} te^{-3t}e^{-st} dt = t \frac{e^{-(s+3)t}}{-(s+3)} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-(s+3)t}}{-(s+3)} dt = \frac{1}{(s+3)^2}; \quad \text{Re}(s) > -3$$

The Laplace transform of a sum is the sum of the Laplace transforms,

$$X_2(s) = \frac{1}{s} - \frac{1}{s+3} + \frac{1}{(s+3)^2} = \frac{4s + 9}{s(s+3)^2}$$

and the region of convergence is the intersection of the 3 regions: $\text{Re}(s) > 0$.

Örnek: Yakınsama Bölgesi - ROC

$$x_3(t) = |t|e^{-|t|}$$

$X_3 =$

$$\frac{2(s^2 + 1)}{(s + 1)^2(s - 1)^2}$$

ROC:

$$-1 < \operatorname{Re}(s) < 1$$

The signal $x_3(t)$ can be written as $te^{-t}u(t) - te^t u(-t)$. The transform of the first is

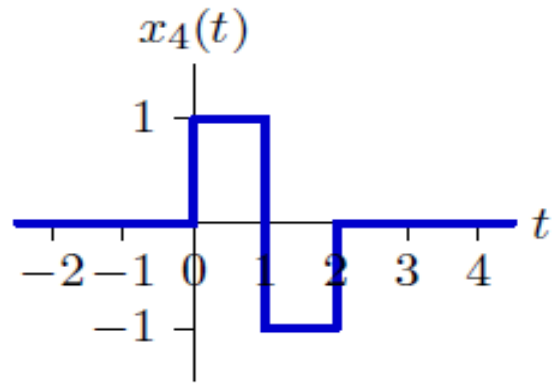
$$\frac{1}{(s + 1)^2} ; \operatorname{Re}(s) > -1$$

and the transform of the second is

$$\frac{1}{(s - 1)^2} ; \operatorname{Re}(s) < 1.$$

Therefore

$$X_3(s) = \frac{1}{(s + 1)^2} + \frac{1}{(s - 1)^2} = \frac{2(s^2 + 1)}{(s + 1)^2(s - 1)^2} ; -1 < \operatorname{Re}(s) < 1.$$



$X_4 =$

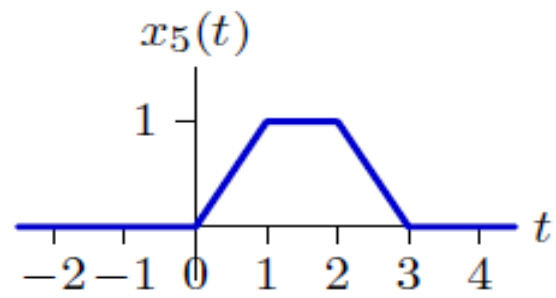
$$\frac{e^{-2s} - 2e^{-s} + 1}{s}$$

ROC:

all s

$$\begin{aligned} X_4(s) &= \int_{-\infty}^{\infty} x_4(t)e^{-st} dt = \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^1 - \left. \frac{e^{-st}}{-s} \right|_1^2 \\ &= -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} = \frac{e^{-2s} - 2e^{-s} + 1}{s} \end{aligned}$$

These integrals all converge for all values of s . Therefore the region of convergence is the entire s -plane.



$X_5 =$

$$\frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}$$

ROC:

all s

Laplace Transform Pairs of $\exp(-at)$

$$f(t) = e^{-at}$$

$$\mathbf{L}[f(t)] = \int_{0^-}^{\infty} e^{-at} \cdot e^{-st} dt = \int_{0^-}^{\infty} e^{-(a+s)t} dt$$

$$= \frac{1}{-(a+s)} \left[e^{-(a+s)t} \right]_{t=0^-}^{t=\infty} = \frac{1}{-(a+s)} [0 - 1] = \frac{1}{s+a}$$

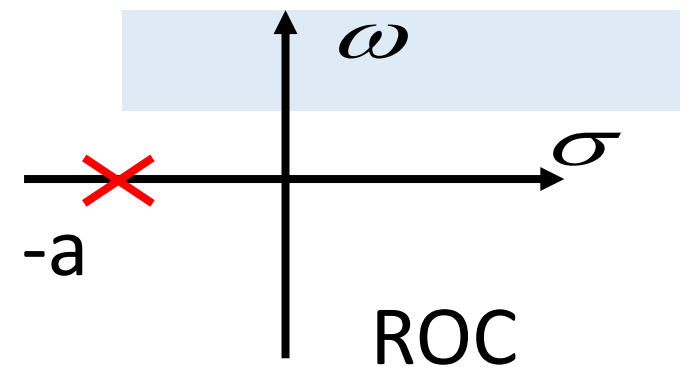
If $\text{Re}[a+s] = a + \sigma > 0$

Time domain: e^{-at}



s-domain: $1/(s+a)$

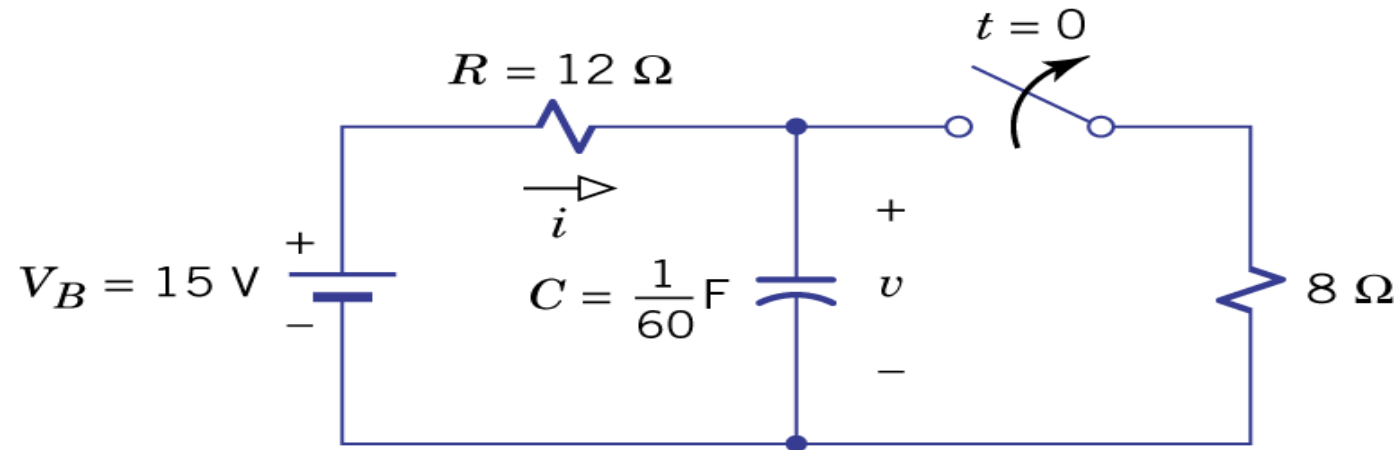
ROC: $\sigma > -a$



Laplace Transform

Circuit Application

Differential Equation



Find $v(t)$

$$v(0^-) = \frac{8}{12 + 8} \times 15\text{ V} = 6\text{ V}$$

$$Ri(t) + v(t) = V_B \quad i(t) = Cv'(t)$$

$$RCv'(t) + v(t) = V_B$$

$$12 \times \frac{1}{60} v'(t) + v(t) = 15$$

$$0.2v'(t) + v(t) = 15$$

$$0.2L[v'(t)] + L[v(t)] = L[15]$$

$$L[15] = \frac{15}{s} \quad V(s) = L[v(t)]$$

$$L[v'(t)] = sL[v(t)] - v(0^-) \\ = sV(s) - 6$$

$$0.2[sV(s) - 6] + V(s) = \frac{15}{s}$$

Differential Equation

$$0.2[sV(s) - 6] + V(s) = \frac{15}{s}$$

$$\Rightarrow V(s) = \frac{1.2 + \frac{15}{s}}{0.2s + 1} = \frac{6s + 75}{s(s + 5)} = \frac{A}{s} + \frac{B}{s + 5}$$

$$sV(s) = \frac{6s + 75}{s + 5} = A + \frac{sB}{s + 5}$$

$$(s + 5)V(s) = \frac{6s + 75}{s} = (s + 5)\frac{A}{s} + B$$

$$V(s) = \frac{15}{s} + \frac{-9}{s + 5}$$

$$\begin{aligned} \mathbf{L}^{-1}[V(s)] \\ = v(t) = 15 - 9e^{-5t} \\ t \geq 0 \end{aligned}$$

$$s = 0 \Rightarrow A = 15$$

$$s = -5 \Rightarrow B = -9$$

Kaynaklar

- Basics of Signals and Systems, Gloria Menegaz, AA 2011-2012
- Signals and Systems, Richard Baraniuk
- Signals and Systems Using MATLAB, Luis F. Chaparro, Department of Electrical and Computer Engineering University of Pittsburgh, Elsevier, 2011.
- Schaum's Outline of Theory and Problems of SIGNALS AND SYSTEMS, 1995
- Sinyaller ve Sistemler Yazar: Michael J. Roberts, Çevirmen: Özgür Salor Durna, Yayınevi: Nobel Akademik Yayıncılık, İlk Baskı Yılı: 2012
- Lecture notes from internet
- https://www3.nd.edu/~b1hu/...09S/Laplace_Transform.ppt
- www.eng.utoledo.edu/~smolitor/courses/.../laplace.ppt

Usage Notes

- These slides were gathered from the presentations published on the internet. I would like to thank who prepared slides and documents.
- Also, these slides are made publicly available on the web for anyone to use
- If you choose to use them, I ask that you alert me of any mistakes which were made and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides.

Sincerely,

Dr. Cahit Karakuş

cahitkarakus@esenyurt.edu.tr